Math 595 Fukaya Categories of Surfaces: Conformal automorphism groups of disks with marked points

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Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$ be the Riemann sphere. The group $\mathrm{GL}(2,\mathbb{C})$ acts on $\widehat{\mathbb{C}}$ by Möbius transformations:

$$\mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}(\widehat{\mathbb{C}}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varphi(z) = \frac{az+b}{cz+d},$$

where in the formula on the right it is to be understood that $\varphi(-d/c) = \infty$ and $\varphi(\infty) = a/c$, unless $c = 0$ in which case $\varphi(\infty) = \infty$. This homomorphism is surjective onto the group of holomorphic automorphisms $\mathrm{Aut}(\widehat{\mathbb{C}})$, and its kernel consists of scalar multiples of the identity matrix. Thus it descends to an isomorphism

$$\mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{Aut}(\widehat{\mathbb{C}}).$$

Now observe that the homomorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C})$ is surjective, since the field $\mathbb{C}$ has the property that every element is a perfect square. The kernel of this map is $\{I, -I\}$, where I denotes the identity matrix, and so we have isomorphisms

$$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\} \cong \mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{Aut}(\widehat{\mathbb{C}}).$$

It is a standard fact that $\mathrm{Aut}(\widehat{\mathbb{C}})$ acts simply transitively on triples of pairwise distinct points in $\widehat{\mathbb{C}}$. This is equivalent to the statement that, for any triple (z_0, z_1, z_2) of pairwise distinct points, there is a unique Möbius transformation φ such that $\varphi(0) = z_0$, $\varphi(1) = z_1$, and $\varphi(\infty) = z_2$. Solving for φ given z_i is an elementary problem.

Consider the open upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$. The closure of $\mathbb{H}$ in $\widehat{\mathbb{C}}$ will be denoted $\overline{\mathbb{H}}$; thus $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$. A Möbius transformation that preserves $\mathbb{H}$ must also preserve its boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Therefore this transformation must be representable by a matrix with real entries. Note that $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R})$ both have two connected components that are distinguished by the sign of the determinant. The identity component of $\mathrm{PGL}(2,\mathbb{R})$ preserves $\mathbb{H}$, while the other component swaps $\mathbb{H}$ with the lower half-plane. The mapping

$$\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R})$$

is an isomorphism onto the identity component. We conclude that there is an isomorphism

$$\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H}).$$

Just as $\mathrm{PGL}(2,\mathbb{C})$ acts simply transitively on triples of pairwise distinct points in $\widehat{\mathbb{C}}$, the group $\mathrm{PGL}(2,\mathbb{R})$ acts transitively on triples of pairwise distinct points in $\mathbb{R} \cup \{\infty\}$. (The arguments can be done in parallel if we think of $\widehat{\mathbb{C}}$ as the projective line over $\mathbb{C}$ and $\mathbb{R} \cup \{\infty\}$ as the projective line over $\mathbb{R}$.)

However, the action of $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H})$ has two orbits: if we orient $\mathbb{R} \cup \{\infty\}$ as the boundary of $\overline{\mathbb{H}}$, there are two possible cyclic orderings of the three points: $z_0 < z_1 < z_2 < z_0$ and $z_0 < z_2 < z_1 < z_0$. The action of $\mathrm{PSL}(2,\mathbb{R})$ preserves this cyclic ordering because it acts on $\overline{\mathbb{H}}$ by orientation-preserving diffeomorphisms. (The nonidentity component of $\mathrm{PGL}(2,\mathbb{R})$, which swaps the upper and lower half-planes, reverses the cyclic ordering.) The conclusion is that $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H})$ acts simply transitively on triples of pairwise distinct points in $\partial\overline{\mathbb{H}}$ with a fixed cyclic ordering.

This has a consequence that if we consider configurations of one or two points on $\partial\overline{\mathbb{H}}$, there is a nontrivial stabilizer subgroup of $\mathrm{Aut}(\mathbb{H})$.

First consider configurations of two distinct points on $\partial\overline{\mathbb{H}}$. There is only one cyclic ordering of two points, so the action of $\mathrm{Aut}(\mathbb{H})$ is transitive. Thus the stabilizers for all configurations are mutually conjugate subgroups, and we might as well just pick a single configuration and compute its stabilizer. So consider the configuration $(z_0 = 0, z_2 = \infty)$. The general form of a Möbius transformation in $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H})$ is

$$\varphi(z) = \frac{az+b}{cz+d}$$

with $a,b,c,d \in \mathbb{R}$ and $ad-bc = 1$ The condition $\varphi(\infty) = \infty$ means $c = 0$, and the condition that $\varphi(0) = 0$ means $b = 0$. Thus $\varphi(z) = ad^{-1}z$. Note that $ad = 1$ means that $ad^{-1} = a^2$. So

$$\varphi(z) = a^2 z \quad (a \in \mathbb{R}^\times)$$

is multiplication by some positive real number a^2. Thus the stabilizer subgroup is isomorphic to the multiplicative group $(\mathbb{R}_{>0}, \cdot)$ of positive real numbers. Via the exponential map this is isomorphic to the additive group $(\mathbb{R}, +)$.

Now consider the configurations of one point on $\partial\overline{\mathbb{H}}$. By transitivity we may as well take this point to be ∞. The condition $\varphi(\infty) = \infty$ again means $c = 0$, so $ad = 1$ and so φ reduces to

$$\varphi(z) = a^2 z + ab \quad (a \in \mathbb{R}^\times, b \in \mathbb{R}).$$

In other words, the stabilizer subgroup is the group of affine linear transformations of $\mathbb{R}$ with positive leading coefficient. This is a two-dimensional connected and simply connected nonabelian Lie group; these properties characterize it up to isomorphism.

So far we have used the upper half-plane model, but we could also use the unit disk model. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, and let $\overline{\mathbb{D}}$ be its closure. To compare $\mathbb{H}$ and $\mathbb{D}$ we must choose a conformal isomorphism between them. For lack of a canonical choice, let us choose

$$f(z) = \frac{z-i}{z+i}.$$

This defines a map $\mathbb{H} \to \mathbb{D}$, with $f(0) = -1$, $f(1) = -i$ and $f(\infty) = 1$.

By general nonsense, the group $\mathrm{Aut}(\mathbb{D})$ is conjugate to $\mathrm{Aut}(\mathbb{H})$ inside $\mathrm{Aut}(\widehat{\mathbb{C}})$:

$$\mathrm{Aut}(\mathbb{D}) = f\,\mathrm{Aut}(\mathbb{H})\,f^{-1}.$$

Recall that the stabilizer of the configuration $(0,\infty)$ in $\partial\overline{\mathbb{H}}$ is $\varphi(z) = \lambda z$ for $\lambda \in \mathbb{R}_{>0}$. After conjugation by f this becomes

$$\psi(z) = \frac{(\lambda+1)z + (\lambda-1)}{(\lambda-1)z + (\lambda+1)},$$

so these are the conformal automorphisms of $\mathbb{D}$ that fix $(-1,1)$ in $\partial\overline{D}$.

Recall that the stabilizer of $\infty \in \partial\overline{\mathbb{H}}$ is $\varphi(z) = \lambda z + \mu$, where $\lambda \in \mathbb{R}_{>0}, \mu \in \mathbb{R}$. After conjugation by f this becomes

$$\psi(z) = \frac{(\lambda + i\mu + 1)z + (\lambda - i\mu - 1)}{(\lambda + i\mu - 1)z + (\lambda - i\mu + 1)}.$$

A conformal automorphism of $\mathbb{H}$ that fixes ∞ is completely determined by where it sends i: with the notation as above, $\varphi(i) = \lambda i + \mu = \tau \in \mathbb{H}$. Since $f(i) = 0$, by the same token, a conformal automorphism of $\mathbb{D}$ that fixes 1 is completely determined by where it sends 0, which is some point $\alpha = f(\tau) \in \mathbb{D}$. Thus we may parametrize the stabilizer of $1 \in \partial\overline{\mathbb{D}}$ by points $\alpha \in \mathbb{D}$. We leave it as an exercise to carry this out explicitly.