Towards the Fukaya category of a surface

We have seen how moduli spaces of boundary pointed disks emerge from the Schwarz-Christoffel problem for polygons in the plane.

Now we move from the plane to a Riemann surface S. In the plane we used straight lines as boundary conditions. Now we use (potentially any) smooth curves.

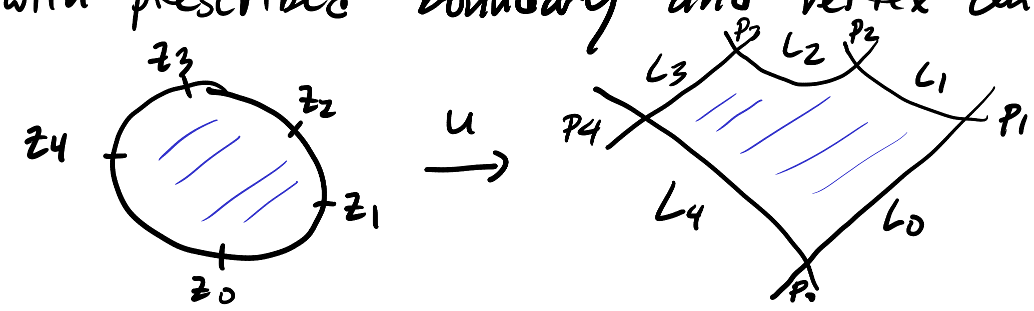
Loose/naïve definition of the Fukaya category $F(S)$ (an A_∞ category over a field k)

* Objects of $F(S)$ are smooth properly embedded curves in S.

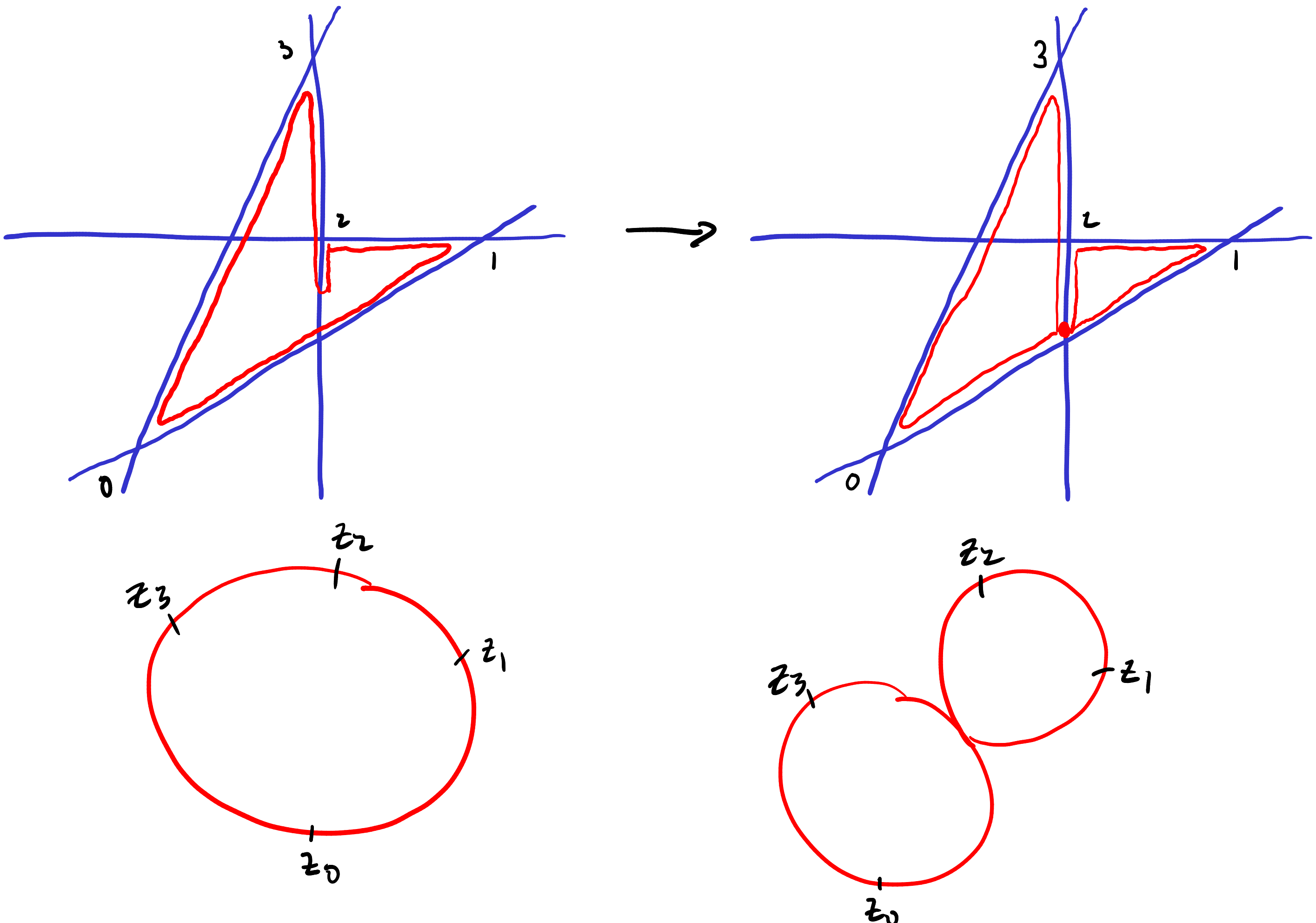
* Given two transversely intersecting curves K, L, the morphism cochain group is a k-vector space spanned by $K \cap L$

$$\hom_{F(S)}(K,L) = \bigoplus_{p \in K \cap L} k \cdot p \qquad \text{(Grading TBD)}$$

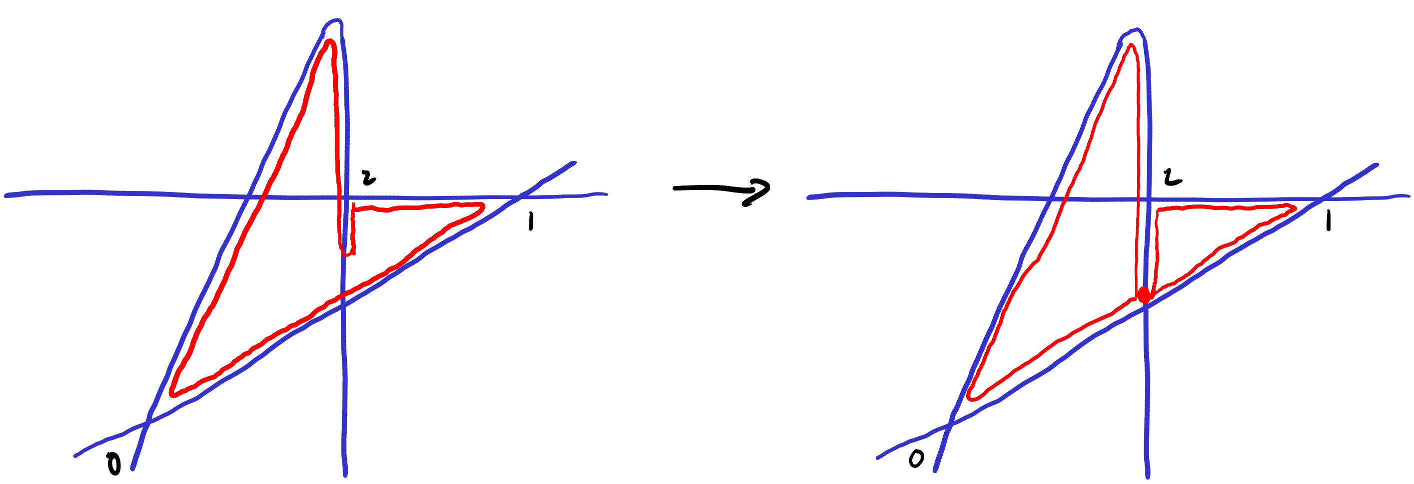
* the A_∞ compositions μ^d "count" holomorphic maps $u: \Sigma \to S$ where $[\Sigma] \in \mathcal{R}^{d+1}$ with prescribed boundary and vertex conditions



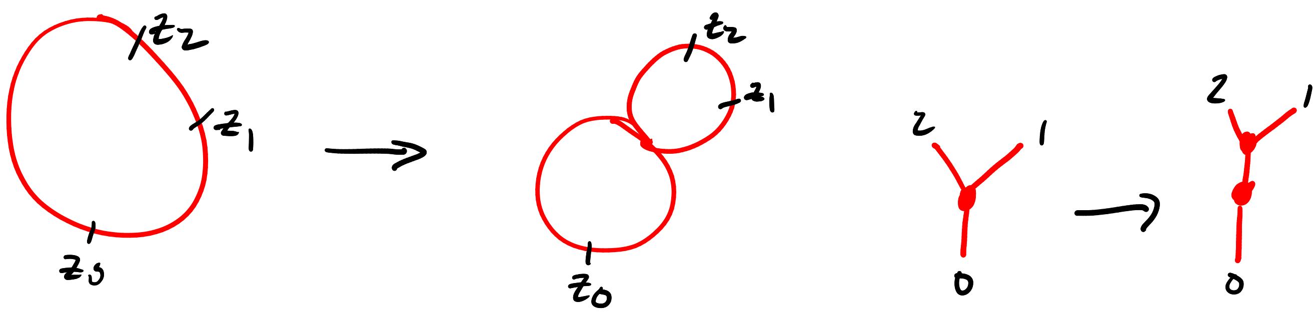
Because the sides of our "polygons" are not required to be "straight" in any sense, some new degeneration phenomena can occur.



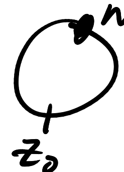
Smooth vertex 3:



The domain appears to degenerate as



This is not one of the degenerations in $\overline{\mathcal{R}}^{d+1}$, because it is "unstable":

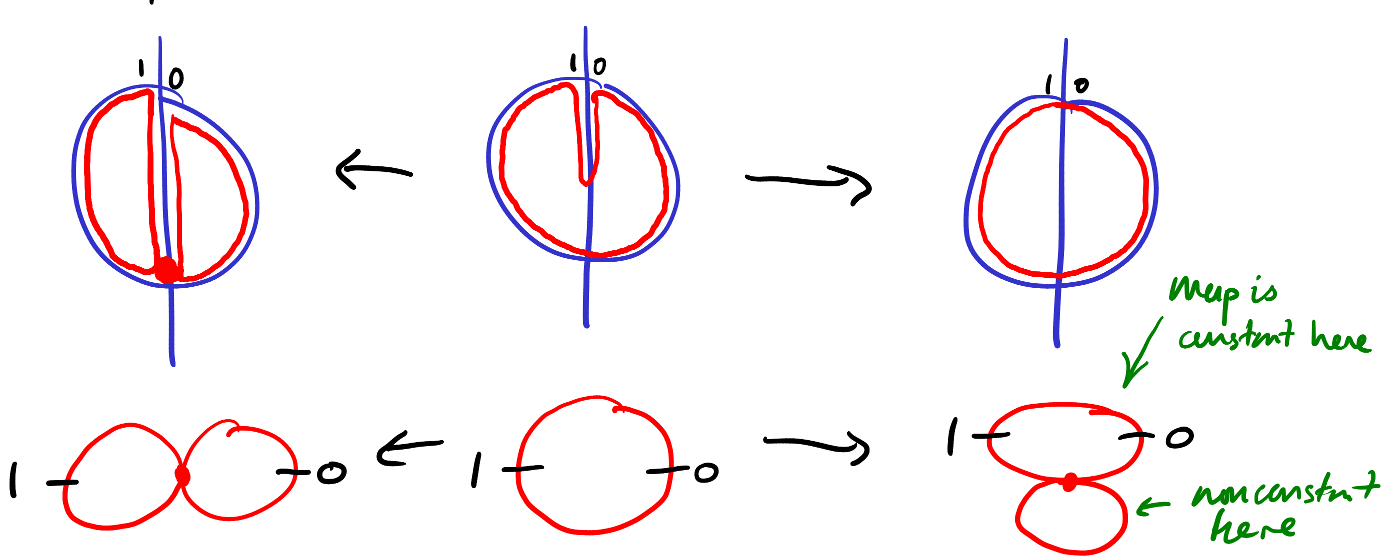
The component 

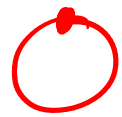
node has automorphism group $\cong \mathbb{R}$, not a finite group.

However, the corresponding holomorphic <u>map</u> to S does have trivial automorphism group.

<u>Lesson</u>: Need to think in terms of holomorphic maps $u : \Sigma \to S$ rather than just subsets of S.

Another phenomenon



The disk with one node  has 2-dimensional automorphism group, but again the map does not have automorphisms.

Also, there is a "ghost component" where the map is constant, but it carries the marked points.

The <u>Stable Map moduli space</u> contains all of these degenerations.