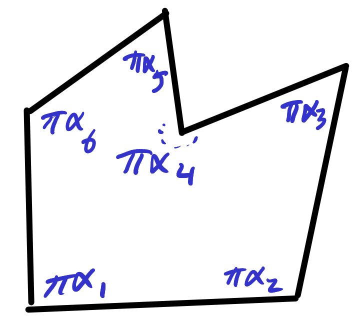
Polygons and moduli spaces of disks.

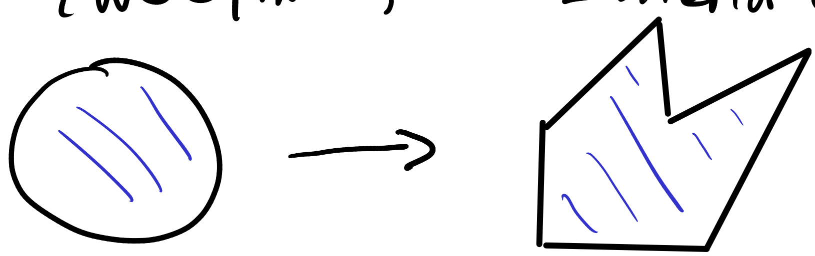
<u>Motivation</u>: polygons in the complex plane

Let P be an n-gon in $\mathbb{C}$ with angles $\pi\alpha_k$ $\alpha_k \in [0,2)$ $(k=1,\dots,n)$



The interior of P is a simply connected domain in $\mathbb{C}$
By the Riemann mapping theorem, there is a biholomorphic map

$$F : D^\circ = \{ w \in \mathbb{C} \mid |w|<1 \} \longrightarrow \text{Interior}(P)$$



(This map will extend continuously but not differentiably over ∂D.)

What is this map? There is a classical answer

<u>Theorem</u> (Schwarz–Christoffel formula, ref: Ahlfors)
There is a map $z = F(w)$ that maps $D^\circ = \{|w|<1\}$ biholomorphically onto Interior(P) of the form

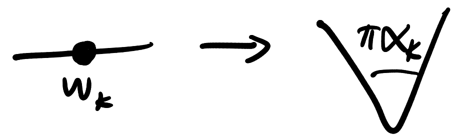
$$F(w) = C \int \prod_{k=1}^{n} (w - w_k)^{\alpha_k - 1} \, dw + C'$$

for some $C, C' \in \mathbb{C}$ and points $w_k \in \partial D$

The points w_k are characterized by

$$\lim_{w \to w_k} F(w) = k\text{-th vertex of } P$$

The map $F(w)$ needs to have a branch-point singularity at w_k in order to map the smooth curve ∂D to the non-smooth curve ∂P

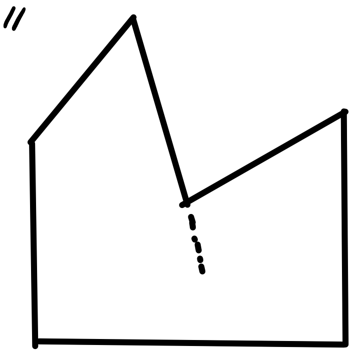


The map $F(w)$ is not unique, but any two choices differ by an element of

$$\mathrm{Aut}(D) \cong PSL(2, \mathbb{R})$$

Now as we deform the polygon P
The map F will need to change. In particular, the points $\{w_k\}_{k=1}^n \subset \partial D$ will move.

If P has a non-convex corner, we can introduce a "cut"

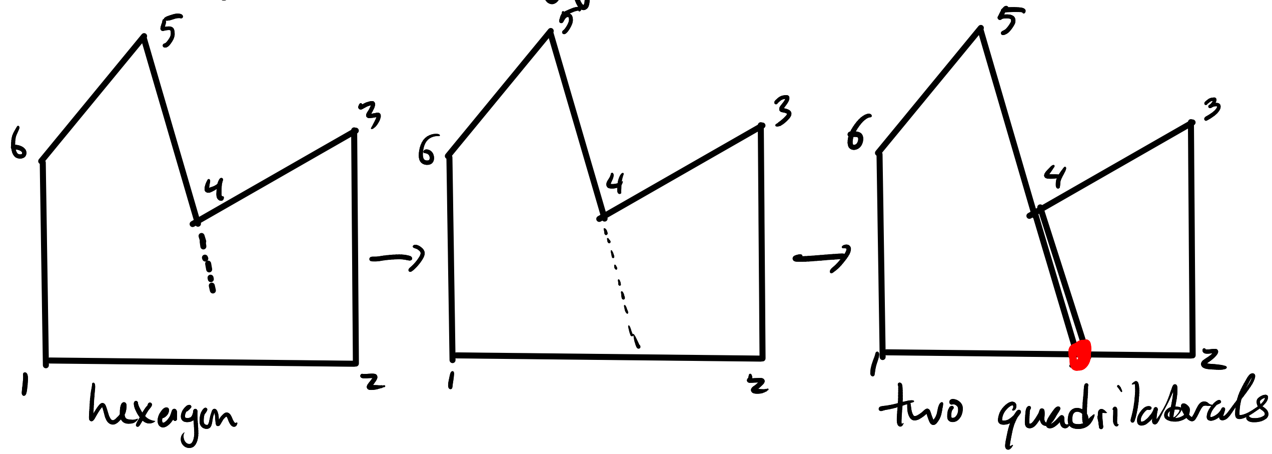


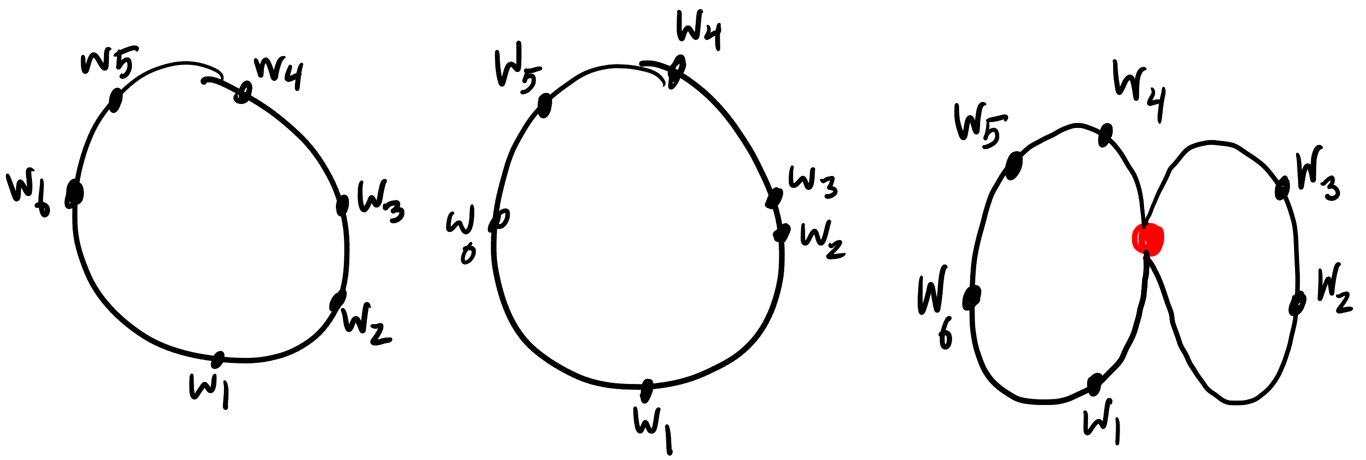
(can be regarded as a new vertex with $\alpha = 2$)

The Riemann mapping theorem still applies, so

There is a family of maps $F : D^\circ \to P \setminus \text{cut}$ parametrized by the length of the cut.
The points $\{w_k\}_{k=1}^n$ move along ∂D in this family

As the cut grows, the polygon breaks into two





From this perspective, the natural domain for the limiting map is a "nodal" disk with two components.

This is made precise by the

$\left\{\begin{array}{l}\text{Deligne-Mumford-Stasheff moduli space of} \\ \text{stable boundary-pointed disks}\end{array}\right\}$

Let $d \geq 2$ Define

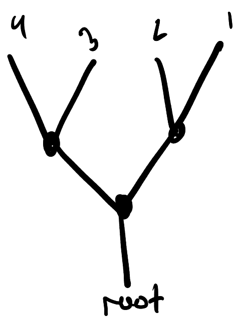
$Conf^{cyc}_{d+1}(\partial D)$ = configurations of $d+1$ distinct points $z_0, z_1, \ldots, z_d$ on ∂D, cyclically ordered w/r/t orientation of ∂D

$$\mathcal{R}^{d+1} = Conf^{cyc}_{d+1}(\partial D) / Aut(D)$$

$\mathcal{R}^{d+1}$ is the "moduli space of disks with $d+1$ marked boundary points", aka boundary-pointed disks.

$\mathcal{R}^{d+1}$ is not compact; it is homeomorphic to a ball of dimension $d-2$.

Now let T be a planar rooted d-leafed tree.



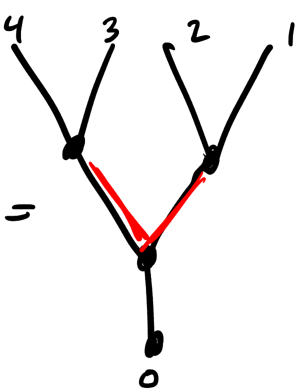
Denote by $V^{int}(T)$ the set of internal vertices (not root or leaf)

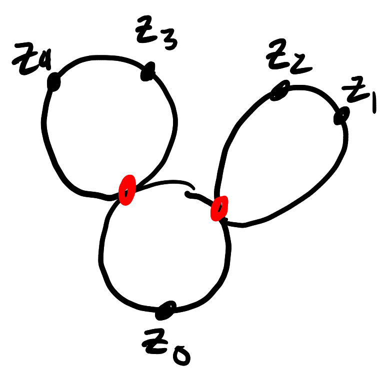
<u>Def</u> T is <u>stable</u> if $\forall v \in V^{int}(T)$ valence$(v) \geq 3$.

Given a stable tree T, define

$$\mathcal{R}^T = \prod_{v \in V^{int}(T)} \mathcal{R}^{\text{valence}(v)}$$

The intended interpretation is that $\mathcal{R}^T$ is a moduli space of nodal disks whose combinatorics is T

$T =$ 

$\mathcal{R}^T = \Big\{$ 

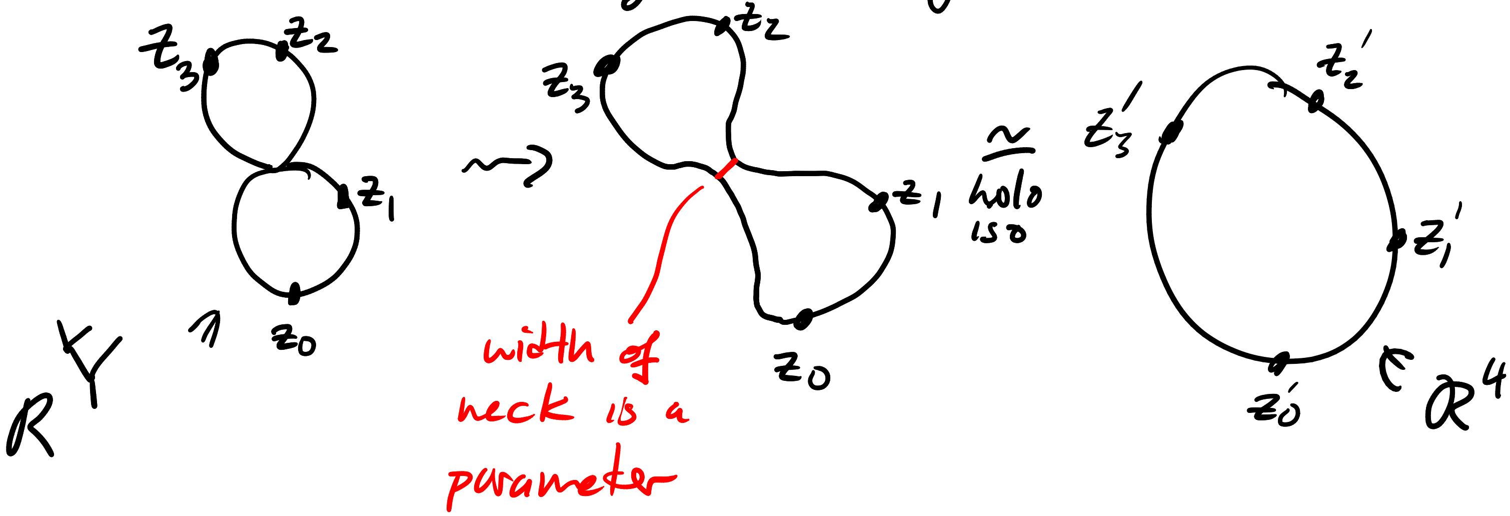
$\Big\} / \text{iso}$

internal edges $\longleftrightarrow$ nodes

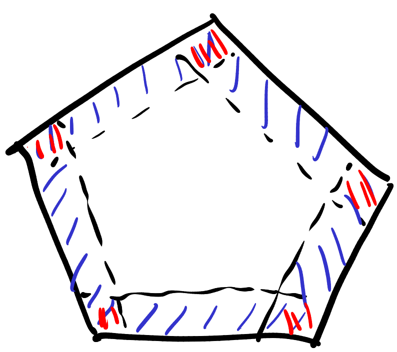
Set theoretically, the Deligne-Mumford-Stasheff space is

$$\overline{\mathcal{R}}^{d+1} = \coprod_{\substack{T \text{ stable} \\ d\text{-leafed} \\ \text{rooted} \\ \text{planar tree}}} \mathcal{R}^T$$

To topologize $\overline{\mathcal{R}}^{d+1}$, we "glue" the boundary strata $\mathcal{R}^T$ onto $\mathcal{R}^{d+1}$ by "smoothing the nodes"



Eg for $\overline{\mathcal{R}}^5$



Shaded area = gluing region.

Now observe: $\overline{\mathcal{R}}^{d+1}$ is stratified by $\mathcal{R}^T$, and the associated poset is the abstract polytope K_d, the Stasheff associahedron.

So $\overline{\mathcal{R}}^{d+1}$ is a "geometric realization" of K_d.