Associahedra form an operad

Let $(\mathcal{C}, \times)$ be a symmetric monoidal category
e.g. (Set, $\times$ = cartesian product)
or $(Ch(k), \times = \otimes_k)$

<u>Def</u> (J.P. May) A (non-symmetric) <u>operad</u> P in $\mathcal{C}$ consists of

- a sequence of objects $P(n) \in \mathrm{Ob}\,\mathcal{C}$ $n \geq 1$
- an element $1_P \in P(1)$
- for each $n \geq 1$ and each sequence $k_1, \dots, k_n \geq 1$ a "composition" map

$$\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) \longrightarrow P(k_1 + \cdots + k_n)$$
$$(\theta, \theta_1, \dots, \theta_n) \mapsto \theta \circ (\theta_1, \dots, \theta_n)$$

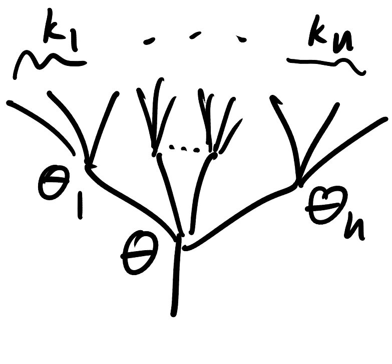
Such that $\theta \circ (1_P, \dots, 1_P) = \theta = 1_P \circ \theta$
and $\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n}))$

$$= (\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n})$$

In terms of planar rooted trees

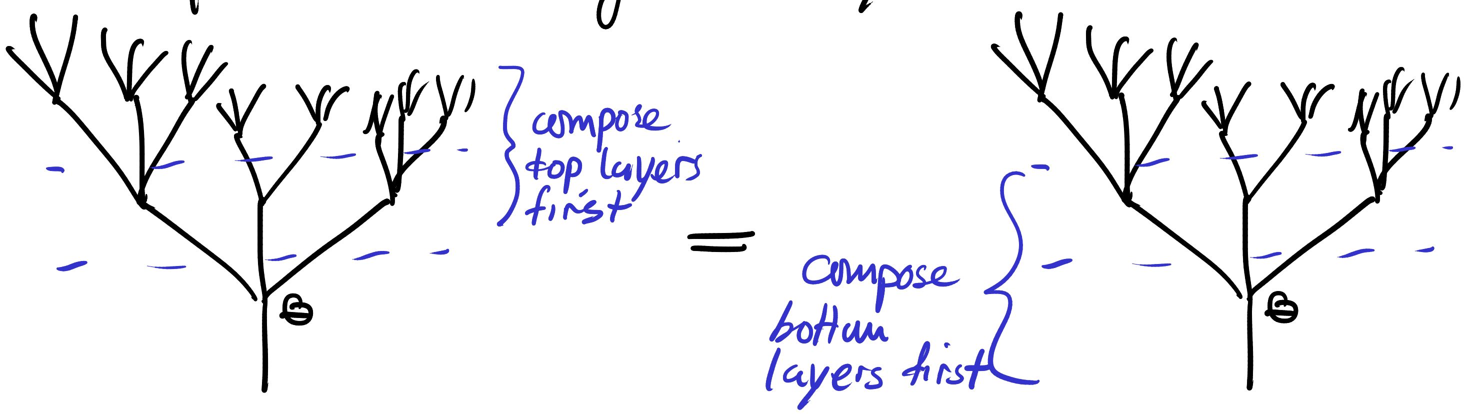
$\circ$: 

$\in P(n) \times P(k_1) \times \cdots \times P(k_n)$

$\mapsto$ 

$\in P(k_1 + \cdots + k_n)$

The operadic associativity axiom says



Given an object $A \in \mathrm{Ob}\,\mathcal{C}$, there is an <u>Endomorphism operad</u> End_A

$$\mathrm{End}_A(n) = \mathrm{Hom}_{\mathcal{C}}(\underbrace{A \times \cdots \times A}_{n}, A) \quad \text{with evident composition}$$

<u>Def</u> A <u>morphism of operads</u> $P \to Q$ is a collection of <u>morphisms</u> $P(n) \to Q(n)$ that preserve compositions and idetity elements.

<u>Def</u> A <u>algebra</u> over an operad P is an object $A \in \mathrm{Ob}\,\mathcal{C}$ together with a morphism of operads
$$\mathrm{act}: P \to \mathrm{End}_A$$

Set theoretically, for each $\theta \in P(n)$ we get a map
$$\mathrm{act}(\theta): \underbrace{A \times \cdots \times A}_{n} \to A$$
That is, an n-ary operation on A.

<u>Ex</u> $\mathcal{C} = \mathrm{Vect}_k$, $\times = \otimes$, $P(n) = k$ for all $n \geq 1$
$$0: k \otimes_k k \otimes \cdots \otimes_k k \to k \quad \text{iso}$$
Algebras over P are associative k-algebras
$$P = \mathrm{Ass}$$

More simple: $\mathcal{C} = \mathrm{Set}$, $\times$ = cart. product

$$P(n) = \{*\} \text{ for all } n \geq 1$$

Algebras = monoids

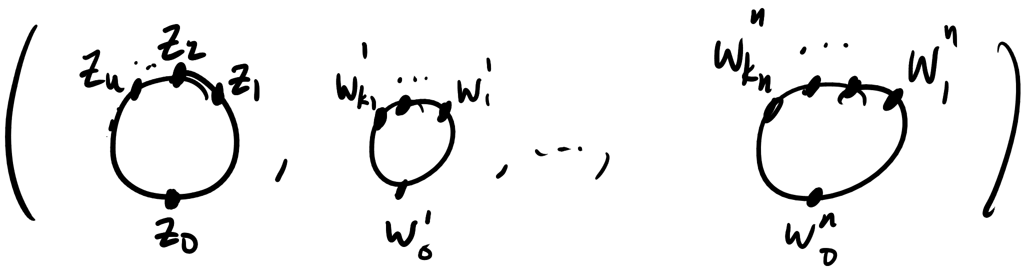
How about $\mathcal{C} = \mathrm{Top}$, $\times$ = product

$(\forall n \geq 1)$ $P(n)$ = some contractible space

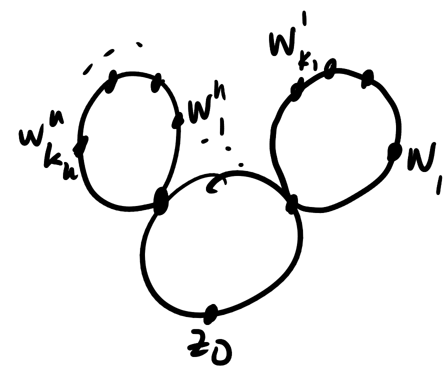
The algebras here are "topological monoids up to homotopy". The A_∞ operad is of this form.

The Deligne–Mumford–Stasheff spaces $\overline{\mathcal{R}}^{d+1}$ carry a natural operad structure where the operad composition is given by "joining at nodes"

Set $P(n) = \overline{\mathcal{R}}^{n+1}$



$\mapsto$



, that is the surface formed by identifying $z_i \sim w_0^i$

Note that this works just as well if inputs are nodal disks.

Also note: open strata get mapped to boundary strata by the composition.

We needed the compactification for this to work.

$P(n) = \overline{\mathcal{R}}^{n+1}$ is an "A_∞-operad" in Top.
Note $P(1) = \{*\}$ by special definition.
We can get an operad in chain complexes by taking the cellular chains $C_*(\overline{\mathcal{R}}^{n+1})$ on $\overline{\mathcal{R}}^{n+1}$ (cell decomposition = stratification)

The operad composition maps take cells to cells, so we get compositions on these complexes.

This operad is "generated" by the top cells $\mathcal{R}^{n+1} \subset \overline{\mathcal{R}}^{n+1}$

An algebra over $\{C_*(\overline{\mathcal{R}}^{n+1})\}_{n \geq 1}$ in chain complexes consists therefor of a chain complex $(A_\bullet, \partial: A_r \to A_{r-1})$

with degree $n-2$ operators $m_n: \underbrace{A \otimes \cdots \otimes A}_{n} \to A \quad n \geq 2$

Converting to cohomological conventions we have

$$(A^\bullet, d: A^r \to A^{r+1})$$

$$\mu^n: A \otimes \cdots \otimes A \to A[2-n]$$

These must satisfy certain relations, it turns out they are the A_∞-associativity equations.

<u>Thus:</u> A_∞-algebras are "the same" as algebras over the operad $C_{-*}(\overline{\mathcal{R}}^{n+1})$ in cochain complexes.