Formal enlargements of A_∞ categories.

We wish to enlarge the Fukaya category for a couple of reasons.

1. To make it a <u>triangulated category</u>, so that we can do homological algebra "in it"

2. To understand in what sense this category may be <u>generated</u> by some collection of objects.

Recall that in ordinary category theory, a category $\mathcal{C}$ has a canonical enlargement

$$\hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \quad \text{(a.k.a. "Presheaves on } \mathcal{C}\text{")}$$

For any object $M \in \mathrm{Ob}\,\mathcal{C}$, there is a functor

$$\mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(-, M) : \mathcal{C}^{op} \to \mathrm{Set}$$

Called a <u>representable functor</u> (represented by M)

The assignment $M \mapsto \mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(-, M)$ extends to a functor

$$\mathcal{Y} : \mathcal{C} \to \hat{\mathcal{C}}$$

Which is fully faithful. $\mathcal{Y}$ is the <u>Yoneda embedding</u>. It allows us to regard $\mathcal{C}$ as a subcategory of $\hat{\mathcal{C}}$.

Constructions in category theory are often formulated by saying that a certain object represents a certain functor.

Given $F : \mathcal{C}^{op} \to Set$, a <u>representation of F</u> is a pair (Z, α) where $Z \in Ob\,\mathcal{C}$ and
$$\alpha : \mathcal{Z} := Hom_{\mathcal{C}}(-, Z) \to F$$
is a natural isomorphism, i.e., an isomorphism in $\hat{\mathcal{C}}$.

<u>Example</u>: given objects Z_0, Z_1 in $\mathcal{C}$, let $F : \mathcal{C}^{op} \to Set$ be the functor
$$F(X) = Hom_{\mathcal{C}}(X, Z_0) \times Hom_{\mathcal{C}}(X, Z_1)$$
(with natural action on morphisms)
An object that represents F (if it exists), is called "the" product $Z_0 \times Z_1$.

In the A_∞ setting, the natural thing to do is to replace Set with the DG category Ch of cochain complexes over k, and consider the category of A_∞ functors $\hat{\mathcal{A}} = Fun_{A_\infty}(\mathcal{A}, Ch)$ for a given A_∞ category $\mathcal{A}$.

We shall opt for an equivalent framework of A_∞-modules over $\mathcal{A}$.

<u>Def</u> Let $(\mathcal{A}, \{\mu^d_{\mathcal{A}}\}_{d \geq 1})$ be an A_∞ category. An A_∞-module $\mathcal{M}$ consists of:

- For each $X \in Ob\,\mathcal{A}$, a graded k-vector space $\mathcal{M}(X)$
- structure maps

$$\mu^d_{\mathcal{M}} : \mathcal{M}(X_{d-1}) \otimes hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}(X_0)[2-d]$$

Satisfying the following variant of the A_∞-associativity eqns.

$$\sum (-1)^{\maltese} \mu_{\mathcal{M}}^{n+1}(\mu_{\mathcal{M}}^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$
$$+ \sum (-1)^{\maltese} \mu_{\mathcal{M}}^{d-m+1}(b, a_{d-1}, \dots, \mu_{\mathcal{A}}^{m}(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0$$
$$\maltese = \sum_{j=1}^{n} (\deg(a_j) - 1)$$

The first equation says $\mu_{\mathcal{M}}^1 \circ \mu_{\mathcal{M}}^1 = 0$, so $\mathcal{M}(X)$ is a cochain complex

For each $a \in \hom_{\mathcal{A}}^i(X_0, X_1)$, $\mu_{\mathcal{M}}^2(-, a) : \mathcal{M}(X_1) \to \mathcal{M}(X_0)$ is a cochain map (up to sign conventions)

Passing to cohomology, $X \mapsto H^{\cdot}(\mathcal{M}(X))$ defines a functor

$$H(\mathcal{A})^{op} \to \text{graded vector spaces.}$$

The collection of all A_∞-modules over $\mathcal{A}$, $\mathrm{mod}(\mathcal{A}) =: \mathcal{Q}$ forms an A_∞ (even DG) category.

$\hom_{\mathcal{Q}}^{p}(\mathcal{M}_0, \mathcal{M}_1)$ consists of collections of maps indexed by d-tuples of objects of $\mathcal{A}$, $(X_0, X_1, \dots, X_{d-1})$

$$t^d : \mathcal{M}_0(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1)$$
$$\to \mathcal{M}_1(X_0)[p-d+1]$$

$$(\mu_{\mathcal{Q}}^1 t)^d(b, a_{d-1}, \dots, a_1)$$

$$= \sum (-1)^{\dagger} \mu_{\mathcal{M}_1}^{n+1}(t^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$+ \sum (-1)^{\dagger} t^{n+1}(\mu_{\mathcal{M}_0}^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$+ \sum (-1)^{\maltese} t^{d-m+1}(b, a_{d-1}, \dots, \mu_{\mathcal{A}}^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$(\mu_{\mathcal{Q}}^2(t_2, t_1))^d(b, a_{d-1}, \dots, a_1)$$

$$= \sum (-1)^{\dagger} t_2^{n+1}(t_1^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n \dots a_1)$$

$$\dagger = \deg(a_{n+1}) + \dots + \deg(a_{d-1}) + \deg(b) - d + n + 1$$

$\mu_{\mathcal{Q}}^d = 0$ for all $d \geq 3$.

The Yoneda embedding now takes the form of an A_∞-functor

$$\mathcal{l}: \mathcal{A} \to mod(\mathcal{A}) = \mathcal{Q}$$

It sets $Y \in Ob\,\mathcal{A}$ to $\mathcal{l}(Y) = \mathcal{Y}$ given by

$$(\mathcal{Y}(X) = hom_{\mathcal{A}}^{\cdot}(X, Y), \quad \mu_{\mathcal{Y}}^d = \mu_{\mathcal{A}}^d).$$

The first component $\mathcal{l}^1 : hom_{\mathcal{A}}(Y_0, Y_1) \to hom_{\mathcal{Q}}(\mathcal{Y}_0, \mathcal{Y}_1)$ sends c to the map

$$(b, a_{d-1}, \dots, a_1) \mapsto \mu_{\mathcal{A}}^{d+1}(c, b, a_{d-1}, \dots, a_1)$$

We can generalize $\mathcal{Y}^1$ to a map, for any A_∞-module $\mathcal{M}$:

$$\lambda : \mathcal{M}(Y) \longrightarrow hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{M})$$

$$\lambda(c)^d(b, a_{d-1}, \dots, a_1) = \mu^{d+1}_{\mathcal{M}}(c, b, a_{d-1}, \dots, a_1)$$

$$\left\{ \begin{array}{c} \text{Analogue in ordinary category theory} \\ Nat(Hom_{\mathcal{C}}(-, Y), F) \simeq FY \\ \alpha \longmapsto \alpha_Y(1_Y) \end{array} \right\}$$

<u>Lemma</u> (Seidel p. 30) λ is a quasi-isomorphism

<u>Corollary</u> $\mathcal{Y}$ is cohomologically full and faithful.

Thus $\mathcal{Y} : \mathcal{A} \to mod(\mathcal{A}) = \mathcal{Q}$ is a fully faithful embedding in the A_∞-sense.

Let $\mathcal{M} \in Ob\, \mathcal{Q}$ be an $\mathcal{A}$-module. A <u>representation</u> for $\mathcal{M}$ is a pair $(Y, [t])$ where $Y \in Ob\, \mathcal{A}$ and $[t] : \mathcal{Y} \to \mathcal{M}$ is an isomorphism in $H^0(\mathcal{Q})$.

Equivalently, there is a $c \in \mathcal{M}^0(Y)$ such that
(i) $\mu^1_{\mathcal{M}}(c) = 0$
(ii) $[t] = [\lambda(c)]$
(iii) for each $X \in Ob\, \mathcal{A}$ the map
$hom_{\mathcal{A}}(X, Y) \to \mathcal{M}(X)$, $b \mapsto (-1)^{deg(b)} \mu^2_{\mathcal{M}}(c, b)$
is a quasi isomorphism.

<u>Direct sum</u> Given A_∞-modules $\mathcal{M}_0$ and $\mathcal{M}_1$ their direct sum has cochain complexes

$$(\mathcal{M}_0 \oplus \mathcal{M}_1)(X) = \mathcal{M}_0(X) \oplus \mathcal{M}_1(X)$$

with obvious structure maps.

If $Y_0, Y_1 \in \mathrm{Ob}\,\mathcal{A}$, we can ask if $\mathcal{Y}_0 \oplus \mathcal{Y}_1$ is representable by an object of $\mathcal{A}$. If it is we denote that object by $Y_0 \oplus Y_1$

<u>Tensor product by cochain complex</u> Let $(Z, d_Z) \in \mathrm{Ob}\,\mathrm{Ch}$ be a cochain complex and let $\mathcal{M}$ be an A_∞-module over $\mathcal{A}$

We define $Z \otimes \mathcal{M}$ by

$$(Z \otimes \mathcal{M})(X) = Z \otimes \mathcal{M}(X)$$

$$\mu^1_{Z\otimes\mathcal{M}}(z \otimes b) = (-1)^{\deg(b)-1} d_Z(z) \otimes b + z \otimes \mu^1_{\mathcal{M}}(b)$$

$$\mu^d_{Z\otimes\mathcal{M}}(z \otimes b, a_{d-1}, \dots, a_1) = z \otimes \mu^d_{\mathcal{M}}(b, a_{d-1}, \dots, a_1)$$

If $Y \in \mathrm{Ob}\,\mathcal{A}$, an object that represents $Z \otimes \mathcal{Y}$ is denoted $Z \otimes Y$.

<u>Shift</u> this is the special case of the above where

$$Z = k[\sigma]$$

$$Z \otimes \mathcal{M} =: \mathcal{M}[\sigma] \quad \text{and} \quad Z \otimes Y =: Y[\sigma]$$

We have $\mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1[\sigma]) \cong \mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1)[\sigma]$

$$\mathrm{Hom}(Y_0[\sigma], Y_1)[\sigma] \cong \mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1)$$

<u>Cones</u> let Y_0, Y_1 be objects of $\mathcal{A}$ and $c \in \hom^0_{\mathcal{A}}(Y_0, Y_1)$ be a degree zero cocycle $\mu^1_{\mathcal{A}}(c) = 0$.

The <u>abstract mapping cone</u> is $\mathcal{C} = \mathcal{C}one(c) \in \mathrm{Ob}\,\mathcal{Q}$

$$\mathcal{C}(X) = \hom_{\mathcal{A}}(X, Y_0)[1] \oplus \hom_{\mathcal{A}}(X, Y_1)$$

$$\mu^d_{\mathcal{C}}((b_0, b_1), a_{d-1}, \ldots, a_1)$$

$$= (\mu^d_{\mathcal{A}}(b_0, a_{d-1}, \ldots, a_1), \mu^d_{\mathcal{A}}(b_1, a_{d-1}, \ldots, a_1) + \mu^{d+1}_{\mathcal{A}}(c, b_0, a_{d-1}, \ldots, a_1))$$

An object of $\mathcal{A}$ that represents $\mathcal{C}$ is denoted $\mathrm{Cone}(c)$.

* <u>Remark</u>: $\mathrm{Cone}(c)$, if it exists, is determined up to <u>canonical</u> isomorphism in $H^0(\mathcal{A})$.
However, if we change c to c' such that $[c] = [c']$ in $H^0(\mathcal{A})$, the objects $\mathrm{Cone}(c)$ and $\mathrm{Cone}(c')$ are <u>NOT</u> canonically isomorphic. (but are isomorphic).

This lack of canonicity for cones in the cohomology categories is one of the deficiencies of the classical theory of triangulated categories, which the theory of DG and A_∞ categories was intended to correct.

$\mathcal{C} = \mathcal{C}one(c)$ is constructed so as to fit into a triangle in $H(\mathcal{Q})$

$$
\begin{tikzcd}
\mathcal{Y}_0 \arrow[rr, "{[\mu^1(c)]}"] & & \mathcal{Y}_1 \arrow[dl, "{[i]}"] \\
& \mathcal{C} \arrow[ul, "{[1]}"', "{[\pi]}"] &
\end{tikzcd}
$$

where $i^1(b_1) = (0, (-1)^{\deg(b_1)} b_1)$ $\quad i^d = 0$ for $d \geq 2$

$\pi^1(b_0, b_1) = (-1)^{\deg(b_0)-1} b_0$ $\quad \pi^d = 0$ for $d \geq 2$

(The morphism $[\pi]$ has degree 1)

An <u>exact triangle in $H(\mathcal{A})$</u> is any diagram

$$
\begin{tikzcd}
Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\
& Y_2 \arrow[ul, "{[1]}"', "{[c_3]}"] &
\end{tikzcd}
$$

that, after applying the Yoneda embedding, $H(\mathcal{A}) \to H(\mathcal{Q})$ becomes isomorphic to such a "cone triangle".

<u>Def</u> A A_∞-category $\mathcal{A}$ is <u>triangulated</u> if it is nonempty $(Ob\, \mathcal{A} \neq \emptyset)$ and

1. every morphism $[c_1] : Y_0 \to Y_1$ in $H^0(\mathcal{A})$ may be completed to an exact triangle
2. for any object Y, there is $\hat{Y}$ s.t. $\hat{Y}[1] \cong Y$ in $H^0(\mathcal{A})$

Condition 1 implies several things

If $Y \in \mathrm{Ob}\,\mathcal{A}$ and $[1_Y] \in \mathrm{Hom}_{H^0(\mathcal{A})}(Y,Y)$
Then the cone of $[1_Y]$ exists in $\mathcal{A}$ by 1.
But this is a null object

$$\begin{tikzcd}
Y \arrow[rr, "{[1_Y]}"] & & Y \arrow[dl, "0"] \\
& 0 \arrow[ul, "{[1]}" , "0"'] &
\end{tikzcd}$$

in the sense that $\mathrm{Hom}_{H(\mathcal{A})}(Y,0) = 0 = \mathrm{Hom}_{H(\mathcal{A})}(0,Y)$

Then $\mathrm{cone}(0 : Y \to 0) = Y[1]$, so (1) implies that shifts exist, and 2 says that the shift $[1]$ is an autoequivalence of $\mathcal{A}$.

But then consider $\mathrm{Cone}(Y_0[-1] \xrightarrow{0} 0 \xrightarrow{0} Y_1)$

this is $Y_0 \oplus Y_1$, so direct sums exist.

<u>Example</u> $\mathcal{Q} = \mathrm{mod}(\mathcal{A})$ is always triangulated.

<u>Prop</u> If $\mathcal{A}$ is triangulated A_∞-category, then $H^0(\mathcal{A})$ is triangulated in the classical sense of Verdier.
If $F : \mathcal{A} \to \mathcal{B}$ is an A_∞-functor between triangulated A_∞-categories, then $H^0(F)$ is an exact functor (preserves distinguished triangles)

Let $\mathcal{B}$ be a triangulated A_∞-category and $\mathcal{A} \subset \mathcal{B}$ a full subcategory. <u>The triangulated subcategory generated by $\mathcal{A}$</u> is the smallest full subcat $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that

- $\mathcal{A} \subset \tilde{\mathcal{B}}$
- $\tilde{\mathcal{B}}$ is triangulated
- $\tilde{\mathcal{B}}$ is closed under isomorphism (if $X_0 \cong X_1$ in $H^0(\tilde{\mathcal{B}})$, then $X_0 \in \mathrm{Ob}\,\tilde{\mathcal{B}} \Leftrightarrow X_1 \in \mathrm{Ob}\,\tilde{\mathcal{B}}$)

If $\tilde{\mathcal{B}} = \mathcal{B}$ then we say <u>$\mathcal{A}$ generates $\mathcal{B}$</u>.

A <u>triangulated envelope</u> for $\mathcal{A}$ is a pair $(\mathcal{B}, F)$ where $\mathcal{B}$ is triangulated, $F : \mathcal{A} \to \mathcal{B}$ is cohomologically full and faithful, and $F(\mathrm{Ob}\,\mathcal{A})$ generates $\mathcal{B}$.

<u>Prop</u> Triangulated envelopes always exist and are unique up to A_∞-quasi-equivalence

We denote by $\mathcal{A}^{tr}$ a triangulated envelope. Then $H^0(\mathcal{A}^{tr})$ is independent of the choice of envelope up to equivalence.

$$D(\mathcal{A}) = H^0(\mathcal{A}^{tr})$$ is the <u>derived category of $\mathcal{A}$</u>; it is a triangulated k-linear category.

One construction of $\mathcal{A}^{tr}$ is to take $\ell : \mathcal{A} \to \mathcal{Q} = \mathrm{mod}(\mathcal{A})$ and take $\mathcal{A}^{tr}$ = subcat of $\mathcal{Q}$ generated by Yoneda modules $\mathcal{Y}$ of $Y \in \mathrm{Ob}\,\mathcal{A}$.

A more concrete construction is often preferred.

<u>Twisted complexes.</u>

I. Additive enlargement $\Sigma\mathcal{A}$

$\mathrm{Ob}\,\Sigma\mathcal{A}$ = formal direct sums $X = \bigoplus_{i\in I} V^i \otimes X^i$

where I is a finite set, $X^i \in \mathrm{Ob}\,\mathcal{A}$,
and V^i are finite-dim graded vector spaces.

$$\hom_{\Sigma\mathcal{A}}\Big(\bigoplus_{i\in I_0} V_0^i \otimes X_0^i,\ \bigoplus_{j\in I_1} V_1^j \otimes X_1^j\Big)$$

$$= \bigoplus_{i,j} \hom_k(V_0^i, V_1^j) \otimes \hom_{\mathcal{A}}(X_0^i, X_1^j)$$

An element is written $(a^{ji})_{j\in I_1,\, i\in I_0}$ with $a^{ji} = \sum_k \phi^{jik} \otimes x^{jik}$

$\mu^d_{\Sigma\mathcal{A}}$ combines $\mu^d_{\mathcal{A}}$ with composition of $\hom(V_k^i, V_{k+1}^j)$
(with appropriate Koszul signs)

$\mathcal{A} \hookrightarrow \Sigma\mathcal{A}$ as $I = \{*\}$ $V^* = k$ $X^* = X$

A <u>pre-twisted complex</u> is a pair (X, δ_X)
where $X \in \mathrm{Ob}\,\mathcal{A}$ and $\delta_X \in \hom^1_{\Sigma\mathcal{A}}(X, X)$

That is, $X = \bigoplus_{i \in I} V^i \otimes X^i \quad \delta_X = (\delta_X^{ji})$

$$\delta_X^{ji} = \sum_k \phi^{jik} \otimes x^{jik} \quad \text{and} \quad \deg(\phi^{jik}) + \deg(x^{jik}) = 1$$

A "naive subcomplex" is a choice of subspace $\tilde{V}^i \subset V^i$ for each i such that δ_X restricts to an endomorphism of $\bigoplus_{i \in I} \tilde{V}^i \otimes X^i$.

A pre-twisted complex is a <u>twisted complex</u> if the following conditions are satisfied.

(a) There is some filtration of X by naive subcomplexes
$X = F^0X \supset F^1X \supset \cdots \; F^nX = 0$
such that the operator on $F^kX/F^{k+1}X$ induced by δ_X is zero. (δ_X is strictly lower triangular)

(b) δ_X satisfies the A_∞-Maurer-Cartan equation

$$\sum_{r=1}^{\infty} \mu^r_{\Sigma A}(\underbrace{\delta_X, \ldots, \delta_X}_{r}) = 0$$

(which is a finite sum because of (a))

The twisted complexes are the objects of an A_∞ category $Tw\,A$. The operations involve "inserting δ in all possible ways"

$$a_1 \in \hom_{TwA}((X_0, \delta_{X_0}), (X_1, \delta_{X_1})) = \hom_{\Sigma A}(X_0, X_1)$$

$$\mu^1_{TwA}(a_1) = \sum_i \mu^{1+i_0+i_1}(\underbrace{\delta_{X_1}, \ldots, \delta_{X_1}}_{i_1}, a_1, \underbrace{\delta_{X_0}, \ldots, \delta_{X_0}}_{i_0})$$

$$\mu^d_{Tw\mathcal{A}}(a_d,\dots,a_1) = \sum_{i_0, i_1 \dots i_d} \mu^{d+i_0+\dots+i_d}_{\Sigma\mathcal{A}}\Big(i_d\,\{\, \delta_{X_d},\dots,\delta_{X_d},\ a_d,\quad i_{d-1}\,\{\, \delta_{X_{d-1}},\dots,\delta_{X_{d-1}},\ a_{d-1},\quad \dots,\quad i_1\,\{\, \delta_{X_1},\dots,\delta_{X_1},\ a_1,\quad i_0\,\{\, \delta_{X_0},\dots,\delta_{X_0} \Big)$$

The Maurer-Cartan equation implies that these operations satisfy the A_∞-associativity equations.

<u>Thm</u>: $\mathcal{A} \hookrightarrow Tw\mathcal{A}$ is a triangulated envelope of $\mathcal{A}$.

<u>Remark</u>: In the DG setting ($\mu^d_{\mathcal{A}} = 0$ for $d \geq 3$) the Maurer-Cartan equation makes sense <u>without</u> the strictly-lower-triangular assumption.

Drinfeld showed that this version is not an invariant of $\mathcal{A}$ up to quasiequivalence. That is, it possible to have $\mathcal{A}$ quasiequivalent to $\mathcal{B}$ but the "two-sided twisted complexes of $\mathcal{A}$" is <u>not</u> quasiequivalent "two-sided twisted complexes of $\mathcal{B}$".

<u>Mapping cone in $Tw\mathcal{A}$</u>: let $c \in \hom_{Tw\mathcal{A}}(Y_0, Y_1)$ be closed degree 0.

$$\mathrm{Cone}(c) = \left(C = Y_0[1] \oplus Y_1,\ \delta_C = \begin{pmatrix} \delta_{Y_0}[1] & 0 \\ -c[1] & \delta_{Y_1} \end{pmatrix} \right)$$