Gradings and Signs on surfaces.

In order to put a $\mathbb{Z}$-grading on our morphism spaces $\hom(L_0, L_1)$, and also in order to work over a field of characteristic $\neq 2$, we need to make some other choices and restrict the class of curves that we consider.

Let S be a Riemann surface, possibly with boundary.

The first choice we must make is a nowhere vanishing C^∞ quadratic differential

$$\eta_S^2 \in \Gamma(S, (T^*S)^{\otimes 2}) \quad \text{i.e. } \eta_S^2 : TS \underset{\mathbb{C}}{\otimes} TS \to \mathbb{C}$$

e.g. if z is local coordinate in $U \subseteq S$ $\quad \eta_S^2 = f\, dz^2$, $f \in C^\infty(U, \mathbb{C})$, f nowhere vanishing

η_S^2 is a complex valued quadratic function on TS

so the subset $\quad \mathcal{E}_S = \{ (p, v) \mid p \in S,\ v \in T_pS,\ \eta_S^2(v) \geq 0 \}$

is an $\mathbb{R}$-subbundle of TS (regarded as a rank 2 $\mathbb{R}$-vector bundle) called the line field.

<u>Remarks</u>: The existence of η_S^2 is equivalent to

$$2c_1(S) = 0 \text{ in } H^2(S; \mathbb{Z})$$

If S is not closed, this condition is always satisfied. If S is closed, say genus $= g$, then this condition is satisfied iff $g = 1$. The Fukaya categories of other surfaces cannot be $\mathbb{Z}$-graded using this method.

Once we have chosen η_S^2, the definition of the Fukaya category *depends* on this choice in a *nontrivial* way. But homotopic choices (connected through space of nowhere-vanishing smooth quadratic differentials) lead to equivalent categories.

The subbundle $\xi_S \subset TS$ is an integrable distribution for dimension reasons.

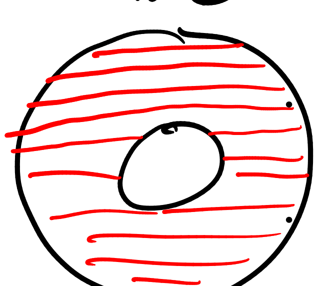
Proof: let X be a local generating vector field for ξ_S. Any two local sections are of the form fX, gX for some local functions f, g.

Then $[fX, gX] = \big(fX(g) - gX(f)\big) \cdot X \in \xi_S$

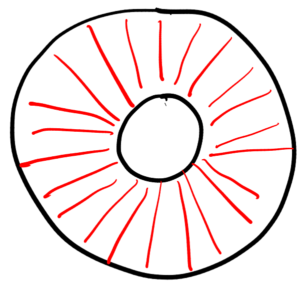
So $\Gamma(S, \xi_S)$ is closed under Lie bracket, and we may apply the Frobenius theorem.

The (unoriented) foliation that integrates ξ_S is a good way to visualize ξ_S or η_S^2.

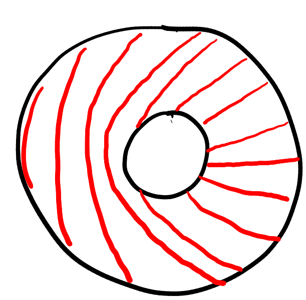
S = Annulus in $\mathbb{C}$



$\eta^2 = dz^2$



$\eta^2 = \dfrac{dz^2}{z^2} = d(\log z)^2$



$\eta^2 = \dfrac{dz^2}{z}$

(ξ not orientable)

The set of homotopy classes of line fields is a torsor for $H^1(S;\mathbb{Z})$. To see this, consider two nowhere vanishing sections $\eta_1^2, \eta_2^2 \in \Gamma(S, (T^*S)^{\otimes 2})$. The ratio $\eta_2^2/\eta_1^2 \in \Gamma(S, \underline{\mathbb{C}})$ is a nowhere vanishing section of the trivial bundle. This may be regarded as a map $S \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is a $K(\mathbb{Z},1)$, the homotopy classes of maps are

$$[S, \mathbb{C}^\times] \simeq [S, K(\mathbb{Z},1)] \simeq H^1(S;\mathbb{Z}).$$

Now consider a curve $L \subset S$, with tangent bundle $TL \subset TS|_L$. An orientation on L amounts to a choice of (homotopy class of) nowhere vanishing section of TL. But $(TL)^{\otimes 2}$ has a <u>canonical</u> class of nowhere vanishing sections. Thus $(T^*L)^{\otimes 2}$ does as well; let's call it ν_L^2.

Then ν_L^2 and $\eta_S^2|_L$ are two nowhere vanishing sections of $(T^*S|_L)^{\otimes 2}$, and their ratio may be regarded as a map

$$\eta_S^2/\nu_L^2 : L \to \mathbb{C}^\times$$

whose homotopy class

$$[\eta_S^2/\nu_L^2] \in [L, \mathbb{C}^\times] \simeq H^1(L;\mathbb{Z})$$

Choosing an orientation on L, we get $H^1(L;\mathbb{Z}) \simeq \mathbb{Z}$, so that this class may be interpreted as a number called the **rotation number** of L with respect to η_S^2 and ξ_S. (This is an instance of the more general concept of the Maslov class of a Lagrangian submanifold.)

A curve L such that $[\eta_S^2/\nu_L^2] = 0 \in H^1(L;\mathbb{Z})$ is called gradable. Any nonclosed arc is gradable, but a null-homotopic curve is <u>never</u> gradable. Otherwise, gradability depends on the curve and the choice of line field ξ_S (η_S^2).

To say that L is gradable is to say that the subbundles TL and $\xi_S|_L$ in $TS|_L$ are homotopic. We would like to choose a specific homotopy.
One way to think of this is as a bundle over $[0,1]\times L$.
Let $\pi : [0,1]\times L \to L$ be projection.

<u>Def</u> A <u>grading</u> on L is an $\mathbb{R}$-subbundle $H \subseteq \pi^*(TS|_L)$ such that

$$H|_{\{0\}\times L} = \xi_S|_L \quad \text{and} \quad H|_{\{1\}\times L} = TL .$$

For fixed $p\in L$, $H|_{[0,1]\times\{p\}}$ is a path of $\mathbb{R}$-subspaces of T_pL. This may also be regarded as a path

$$H(t,p) : [0,1] \longrightarrow \mathbb{RP}(T_pS)$$

$$H(0,p) = \xi_S|_p \qquad H(1,p) = T_pL$$

* Let us measure rotation in $\mathbb{RP}(T_pS)$ so that Closed loops are rotation through $n\pi$ for $n\in\mathbb{Z}$
* positive sense is determined by orientation of S

Let L_1 and L_2 be two curves with gradings H_1 and H_2. Let $p \in L_1 \cap L_2$ be a transverse intersection point.

We can get a path in T_pS from T_pL_1 to T_pL_2 using $H_2(t,p)$ and $H_1(1-t,p)$.

Let $\Theta \in \mathbb{R} \setminus \pi\mathbb{Z}$ be the total angle through which this path rotates.

The absolute index of p is given by

$$i(p) = \lfloor \Theta/\pi \rfloor + 1 \in \mathbb{Z}.$$

Next we consider the problem of signs. This is a very tricky problem and we will use some surprisingly heavy abstraction to tame it.

Let k be a field. The groupoid of k-lines $\mathrm{Line}(k)$ is the category whose objects are one-dimensional k-vector spaces, and whose morphisms are isomorphisms of such vector spaces.

Every morphism in $\mathrm{Line}(k)$ is invertible, so $\mathrm{Line}(k)$ is a groupoid.

Since any two objects in $\mathrm{Line}(k)$ are isomorphic, a skeleton for $\mathrm{Line}(k)$ consists of any single k-line, say k itself, with all its automorphisms.

This remark shows that $\mathrm{Line}(k)$ is equivalent as a category to the multiplicative group $k^\times = k \setminus \{0\}$ regarded as a category with one object.

We will be particularly interested in $\mathrm{Line}(\mathbb{R})$. We can form a certain quotient $\overline{\mathrm{Line}}(\mathbb{R})$

$$\mathrm{Hom}_{\overline{\mathrm{Line}}(\mathbb{R})}(V,W) = \mathrm{Hom}_{\mathrm{Line}(\mathbb{R})}(V,W)/\sim$$

where if $f,g : V \to W$ are two isomorphisms of $\mathbb{R}$-lines, we declare $f \sim g$ iff $\exists\ \lambda \in \mathbb{R}, \lambda > 0$, such that $f = \lambda g$.

The set $\mathrm{Hom}_{\overline{\mathrm{Line}}(\mathbb{R})}(V,W)$ always has two elements

There is an isomorphism of groups $\mathrm{Hom}_{\overline{\mathrm{Line}}(\mathbb{R})}(\mathbb{R},\mathbb{R}) \cong \{\pm 1\}$

The category $\overline{\mathrm{Line}}(\mathbb{R})$ is equivalent to the group $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ regarded as a category w/ one object.

Slogan: The signs we are so worried about are "just" morphisms in $\overline{\mathrm{Line}}(\mathbb{R})$

<u>k-normalization</u>. We need to relate $\mathrm{Line}(\mathbb{R})$ to our coefficient field k. We construct a functor

$$|\cdot|_k : \mathrm{Line}(\mathbb{R}) \to \mathrm{Line}(k) \quad \text{called } \underline{k\text{-normalization}}.$$

Let V be an $\mathbb{R}$-line. There are exactly two orientations of V, let's call them r_1 and r_2

Define $|V|_k = (k\cdot r_1 \oplus k\cdot r_2)/\langle r_1+r_2\rangle$

This is the quotient of the 2-dimensional k-vector space spanned by the orientations, modulo the relation that their sum is zero, so it is a k-line.

Any isomorphism $f: V\to W$ of $\mathbb{R}$-lines maps orientations to orientations, and it maps opposite orientations to opposite ones, so it induces a map

$$|f|_k : |V|_k \to |W|_k$$ which is an isomorphism of k-lines.

Thus we get a functor $|\cdot|_k : Line(\mathbb{R}) \to Line(k)$

Observe that $|\cdot|_k$ factors through $\overline{Line}(\mathbb{R})$

$$\begin{tikzcd} Line(\mathbb{R}) \arrow[rr, "|\cdot|_k"] \arrow[dr, "\pi"'] & & Line(k) \\ & \overline{Line}(\mathbb{R}) \arrow[ur, "|\cdot|_k"'] & \end{tikzcd}$$

So $|\cdot|_k$ is really something about orientations and signs.

Remark: In terms of groups, the diagram looks like

$$\begin{tikzcd} \mathbb{R}^\times \arrow[rr] \arrow[dr, "\frac{x}{|x|}"'] & & k^\times \\ & \{\pm 1\} \arrow[ur] & \end{tikzcd}$$

Tensor products of lines. $\mathrm{Line}(\mathbb{R})$ is symmetric monoidal under tensor product, and every object V has monoidal inverse $V^* = \mathrm{Hom}_{\mathrm{Vect}}(V, \mathbb{R})$

For each $n \in \mathbb{Z}$ there is an endo functor

$$(-)^{\otimes n} : \mathrm{Line}(\mathbb{R}) \to \mathrm{Line}(\mathbb{R})$$

Since products of positive numbers are positive, this descends to a functor

$$(-)^{\otimes n} : \overline{\mathrm{Line}}(\mathbb{R}) \to \overline{\mathrm{Line}}(\mathbb{R})$$

There is also a constant functor $C_{\mathbb{R}} : \overline{\mathrm{Line}}(\mathbb{R}) \to \overline{\mathrm{Line}}(\mathbb{R})$ that sends every object to $\mathbb{R}$ and every morphism to the identity.

<u>Proposition</u> (Even tensor powers are canonically trivial)

If n is even then $(-)^{\otimes n}$ and $C_{\mathbb{R}}$ are isomorphic as functors $\overline{\mathrm{Line}}(\mathbb{R}) \to \overline{\mathrm{Line}}(\mathbb{R})$.

<u>Proof</u> Let V be an $\mathbb{R}$-line, and let r_1 and r_2 be the two orientations of V, represented by vectors v_1 and $v_2 = -v_1$ respectively.

Because n is even, $v_1^{\otimes n} = v_2^{\otimes n}$ in $V^{\otimes n}$

Let $\eta_V \in \mathrm{Hom}_{\overline{\mathrm{Line}}(\mathbb{R})}(V^{\otimes n}, \mathbb{R})$ be the equivalence class of isomorphism taking $v_1^{\otimes n} = v_2^{\otimes n}$ to $1 \in \mathbb{R}$

Then η is a natural isomorphism from $(-)^{\otimes n}$ to $C_{\mathbb{R}}$ $\boxtimes$

<u>Proposition</u> If n is odd then $(-)^{\otimes n}$ is isomorphic to the identity functor.

Now let L_1 and L_2 be two graded curves that intersect transversely at p. We have defined an absolute index $i(p) \in \mathbb{Z}$.
If we regard p as a morphism from L_1 to L_2
We now define the orientation line at p

$$o(p) = (T_p L_2)^{\otimes i(p)} \quad \text{as object of } \mathcal{L}ine(\mathbb{R})$$

That is, we take the tangent space to the target object (L_2) raised to the index.

If $i(p)$ is even then $o(p) \cong \mathbb{R}$ canonically
If $i(p)$ is odd then $o(p) \cong T_p L_2$ canonically.

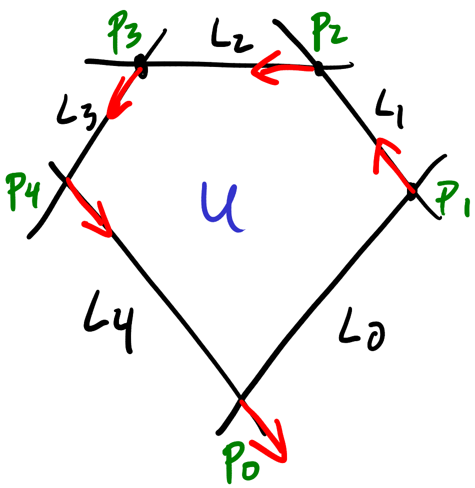
The proper definition of $hom(L_1, L_2)$ is this:
For transversely intersecting graded curves L_1 and L_2 there is a graded k-vector space

$$hom^{\bullet}(L_1, L_2) = \bigoplus_{n \in \mathbb{Z}} hom^n(L_1, L_2)$$

$$hom^n(L_1, L_2) = \bigoplus_{\substack{p \in L_1 \cap L_2 \\ i(p) = n}} |o(p)|_k = \bigoplus_{\substack{p \in L_1 \cap L_2 \\ i(p) = n}} |(T_p L_2)^{\otimes n}|_k$$

This affects the definition of μ^d as follows.

μ^d counts disks, and the boundary orientation of the disk induces orientations on the curves (at least locally near an intersection point)



"apply boundary orientation of disk to target object.

$\hom(L_0,L_1) \geq |o(p_1)|_k = |(T_{p_1}L_1)^{i(p_1)}|_k$

$\hom(L_1,L_2) \geq |o(p_2)|_k$

$\hom(L_2,L_3) \geq |o(p_3)|_k$

$\hom(L_3,L_4) \geq |o(p_4)|_k$

$\hom(L_0,L_4) \geq |o(p_0)|_k = |(T_{p_0}L_4)^{i(p_0)}|_k$

A disk with inputs $p_1,\dots,p_d$ and output p_0 thus gives rise to a unique morphism in $\underline{\mathrm{Line}}(\mathbb{R})$

$$o(u): o(p_d)\otimes o(p_{d-1})\otimes\cdots\otimes o(p_1) \to o(p_0)$$

(match boxes from picture above)

The k-normalization is the contribution to μ^d

$$|o(u)|_k : |o(p_d)\otimes\cdots\otimes o(p_1)|_k \longrightarrow |o(p_0)|_k$$

$$\begin{array}{ccc} |o(u)|_k : |o(p_d)\otimes\cdots\otimes o(p_1)|_k & \longrightarrow & |o(p_0)|_k \\ \wr\| & & \cap \\ |o(p_d)|_k\otimes\cdots\otimes|o(p_1)|_k & & \hom(L_0,L_d) \\ \cap & \nearrow & \\ \hom(L_{d-1},L_d)\otimes\cdots\otimes\hom(L_0,L_1) & & \end{array}$$

It is also a fact that the polygon is rigid iff
$i(p_0) = i(p_1)+\cdots+i(p_d)+2-d$