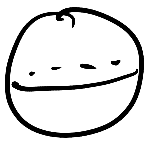
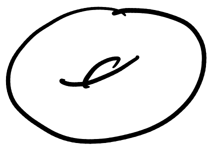
Math 595 FSC Fall 2022
Fukaya Categories of Surfaces

For our purposes, Surfaces are orientable 2-dimensional manifolds

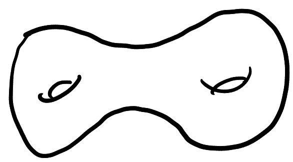
Compact w/o boundary:



Sphere
$g=0$



Torus
$g=1$

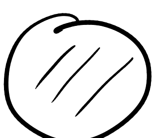


genus 2
$g=2$

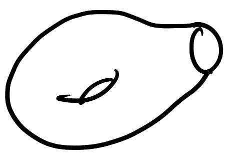
...

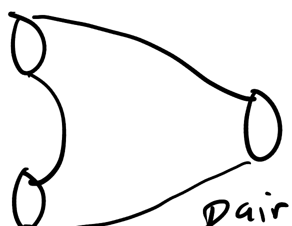
S_g
genus g
surface

We shall also consider surfaces that have boundary, obtain by removing open disks from the above, such as



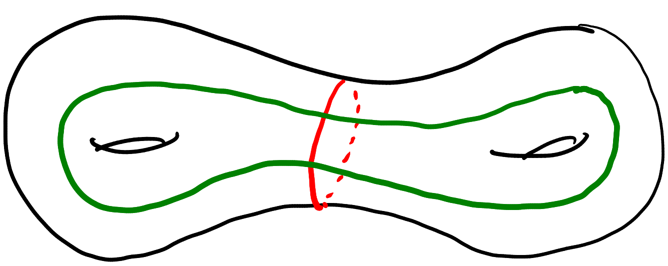
Disk





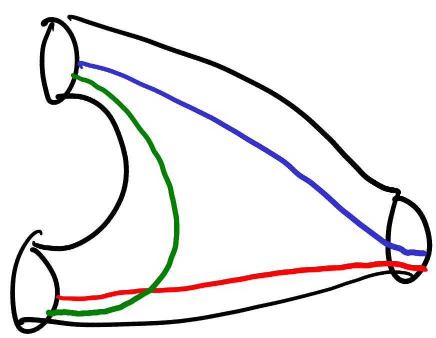
pair of pants

How to probe topology of surfaces? Curves



Closed loops

(self-intersections allowed)



Arcs (when $\partial S \neq \phi$)

The basic invariants of a surface are built from curves π_1 and H_1

$\pi_1(S,p)$ = homotopy classes of loops based at p (nonabelian group, functorial w.r.t. based maps)

$H_1(S) = H_1(S;\mathbb{Z})$ = homology classes of 1-cycles ie. formal sums of oriented loops. (abelian group, functorial w.r.t. all maps)

If S is closed of genus g

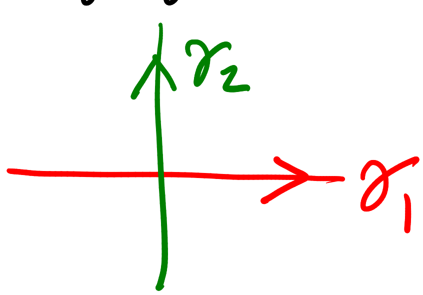
$$\pi_1(S,p) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid \prod_i a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle$$

$$H_1(S) = \mathbb{Z}\langle a_1, b_1, \dots, a_g, b_g \rangle = \mathbb{Z}^{2g}$$

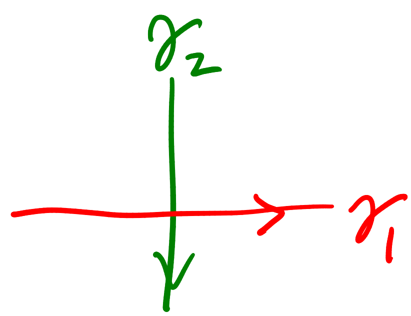
[Recall S = $4g$-gon with sides identified]

More structure: on $H_1(S)$ there is an intersection pairing $(-,-) : H_1(S) \otimes H_1(S) \to \mathbb{Z}$ (once we choose an orientation of S)

For two transversely intersecting loops γ_1 and γ_2 $(\gamma_1, \gamma_2) = \sum_{p \in \gamma_1 \cap \gamma_2} \pm 1$, where the sign is determined locally by the rule



positive



negative

This is well-defined on $H_1(S)$ and is skew-symmetric

$$(\gamma_1, \gamma_2) = -(\gamma_2, \gamma_1)$$

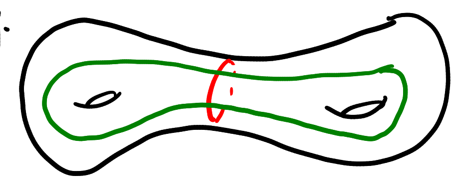
It is also nondegenerate (consequence of Poincaré duality)

We call (γ_1, γ_2) the <u>algebraic intersection number</u>

There is also a <u>geometric intersection number</u> (not homology invariant)

$\mathrm{geo}(\gamma_1, \gamma_2) = \min |\gamma_1 \cap \gamma_2|$ as γ_1 and γ_2 vary in their free homotopy classes.

Clearly $\mathrm{geo}(\gamma_1, \gamma_2) \geq |(\gamma_1, \gamma_2)|$ but inequality may be strict. e.g.



Mapping class group $MCG(S) = \underbrace{\mathrm{Homeo}^+(S)}_{\text{orientation-preserving homeomorphisms}} / \underbrace{\mathrm{Homeo}_0(S)}_{\text{connected component of identity map.}}$

(An important group in low-dim. topology)

$MCG(S)$ acts on $H_1(S)$, preserves $(-,-)$

$$MCG(S) \longrightarrow \mathrm{Aut}\left(H_1(S), (-,-)\right) \simeq Sp(2g, \mathbb{Z})$$

This surjective but not injective

kernel = "Torelli group"

The Fukaya Category Fuk(S) is a certain categorification of the group $H_1(S)$ with the skew pairing $(\cdot,\cdot)$

* Detects geometric intersection numbers

* $MCG \to \mathrm{Auteq}(\mathrm{Fuk}(S))$ is injective

Categorification

$$\begin{aligned}
Ab = \text{abelian groups} &\longrightarrow \mathbb{N} = \{0,1,2,\dots\} \\
M &\longrightarrow \mathrm{rk}\, M = m \\
M \oplus N &\longrightarrow m+n \\
M \otimes N &\longrightarrow m \cdot n \\
\mathrm{Cone}(f: N \to M) &\longrightarrow m-n
\end{aligned}$$

Given some system of numbers eg. $\{(\gamma_1,\gamma_2) \mid \gamma_i \in H_1(S)\}$ Can we find groups (or vector spaces) whose ranks are these numbers, and which is coherent in the sense that the natural relations between the numbers are witnessed by exact sequences of groups/vector spaces?

Given two oriented curves $\gamma_1, \gamma_2 \subseteq S$, we seek* to define a cochain complex $CF^{\cdot}(\gamma_1,\gamma_2)$ such that $\chi(CF^{\cdot}(\gamma_1,\gamma_2)) = (\gamma_1,\gamma_2)$ but $\mathrm{rk}\, H(CF^{\cdot}(\gamma_1,\gamma_2)) = \mathrm{geo}(\gamma_1,\gamma_2)$

The curves γ_1,γ_2 should be objects in a category where $CF^{\cdot}(\gamma_1,\gamma_2)$ is the space of morphisms $\gamma_1 \to \gamma_2$

* As we shall see, this is only possible under certain conditions, but this is the driving idea.