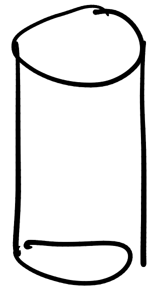
First examples of computations in $F(S)$

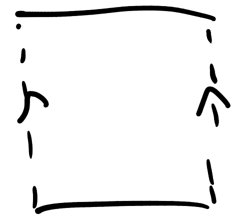
We fix a field k of characteristic 2 (eg. $\mathbb{F}_2$) to avoid signs. We also work without gradings.

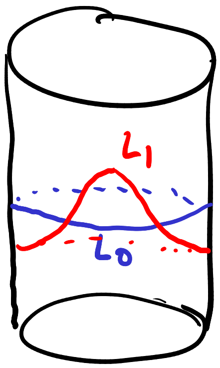
S = cylinder = $S^1 \times [0,1]$

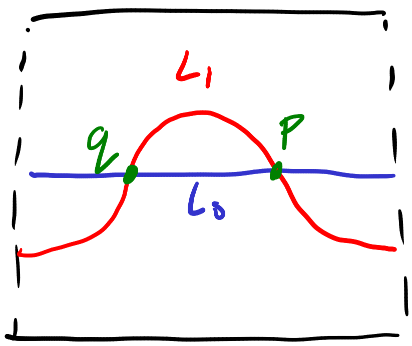
two objects L_0, L_1



or

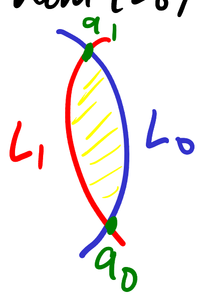






$\hom(L_0, L_1) = \langle p, q \rangle$
2-dimensional k-vector space

The only operation is $\mu^1 : \hom(L_0, L_1) \to \hom(L_0, L_1)$ that counts bigons



q_1 = input
q_0 = output.

There are two such bigons in the picture, and they are rigid modulo $\mathbb{R} \cong \mathrm{Aut}(\mathbb{H}, (0, \infty))$.

Both have input p and output q.

So $\mu^1(p) = q + q = 0$ (char = 2)
$\mu^1(q) = 0$ (no bigons with input q)

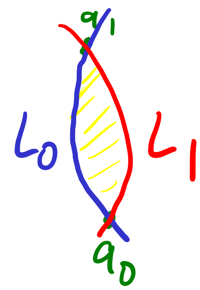
Thus $\mu^1 = 0$ and $\mu^1 \circ \mu^1 = 0$ so we have homology isomorphic to the complex itself.

$$H(\hom(L_0, L_1), \mu^1) = \hom(L_0, L_1) = \langle p, q \rangle$$

2-dim k-v.s.

Next example: same but roles of L_0 and L_1 swapped

$\hom(L_1, L_0) = k\langle p, q \rangle$ again, but now we look for bigons like



There are still two bigons, but now the input is q and the output is p.

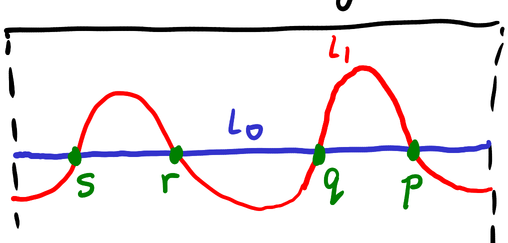
$\mu^1(q) = p + p = 0$ (char 2)

$\mu^1(p) = 0$ (no bigons with input p)

So again $H(\hom(L_1, L_0), \mu^1) = \langle p, q \rangle$

Observe that this computation is evidently "dual" to the one considered before.

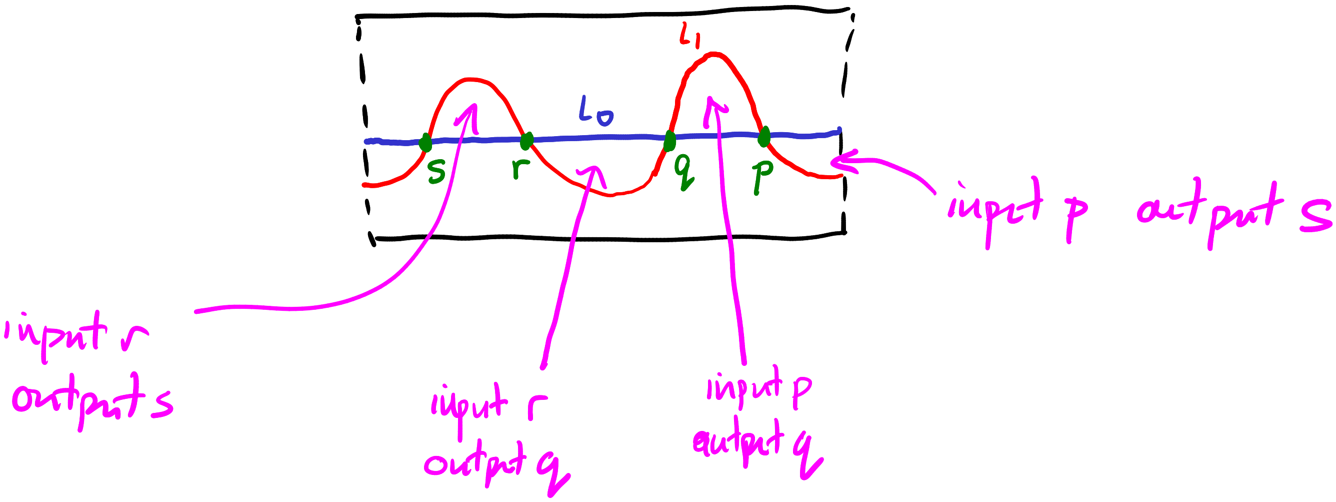
Next, same L_0 but change L_1:



$\hom(L_0, L_1) = \langle p, q, r, s \rangle$

4-dim k-v.s.

This looks locally like the first case.
Now there are 4 bigons



Thus $\mu^1(p) = q+s$ Still have $\mu^1 \circ \mu^1 = 0$
$\mu^1(r) = q+s$ but $\mu^1 \neq 0$.
$\mu^1(q) = 0$
$\mu^1(s) = 0$ Note $\mu^1(p+r) = q+s+q+s = 0$ (Chr 2)

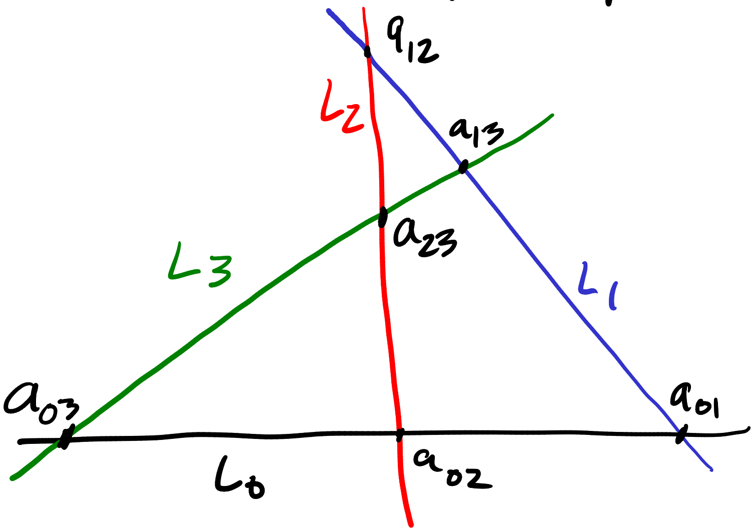
$\ker(\mu^1) = \text{Span}\{q, s, p+r\}$ 3-dim

$\text{Im}(\mu^1) = \text{Span}\{q+s\}$ 1-dim.

So $H(\hom(L_0, L_1), \mu^1) = \dfrac{\ker \mu^1}{\text{Im}\, \mu^1} = \dfrac{\langle q, s, p+r\rangle}{\langle q+s\rangle} = \langle [p+r], [q]=[s]\rangle$
2-dim.

In all cases, we got 2-dim cohomology
This is not an accident, it represents the cohomology $H^*(S^1; k)$ of $S^1 \simeq L_0 \simeq L_1$

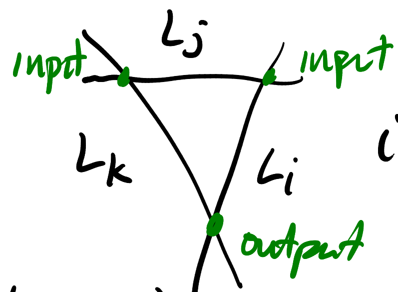
For the next set of examples, let S = disk (or plane)



consider $\hom(L_i, L_j)$ for $i<j$

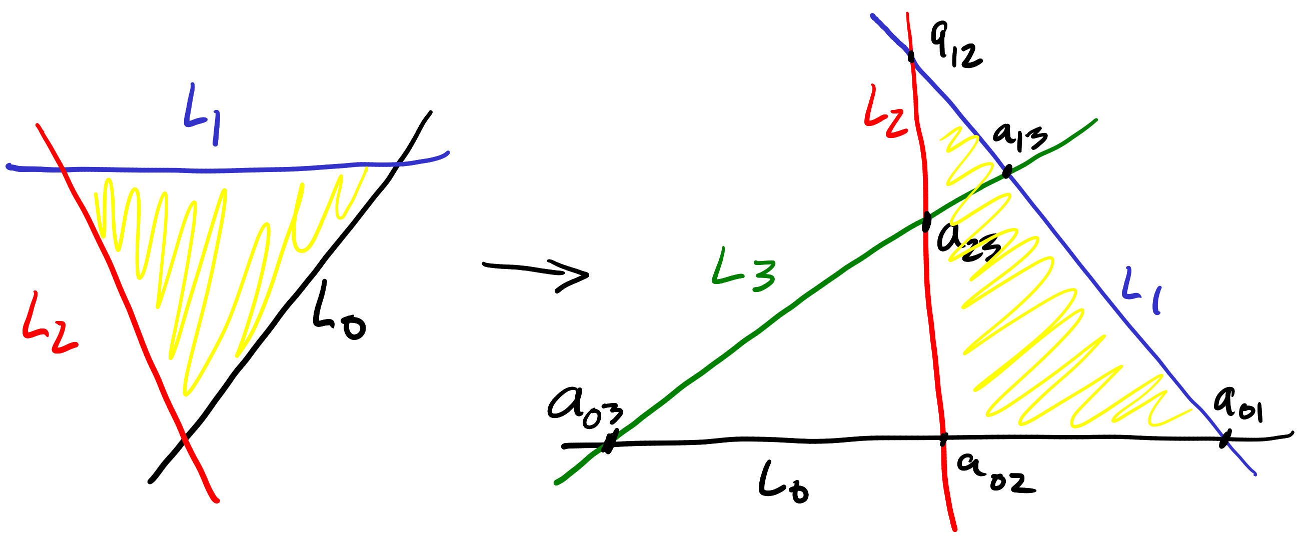
$$\hom(L_0, L_1) = \langle a_{01} \rangle$$
$$\hom(L_0, L_2) = \langle a_{02} \rangle$$
$$\hom(L_0, L_3) = \langle a_{03} \rangle$$
$$\hom(L_1, L_2) = \langle a_{12} \rangle$$
$$\hom(L_1, L_3) = \langle a_{13} \rangle$$
$$\hom(L_2, L_3) = \langle a_{23} \rangle$$

Let's compute μ^2, counting triangles

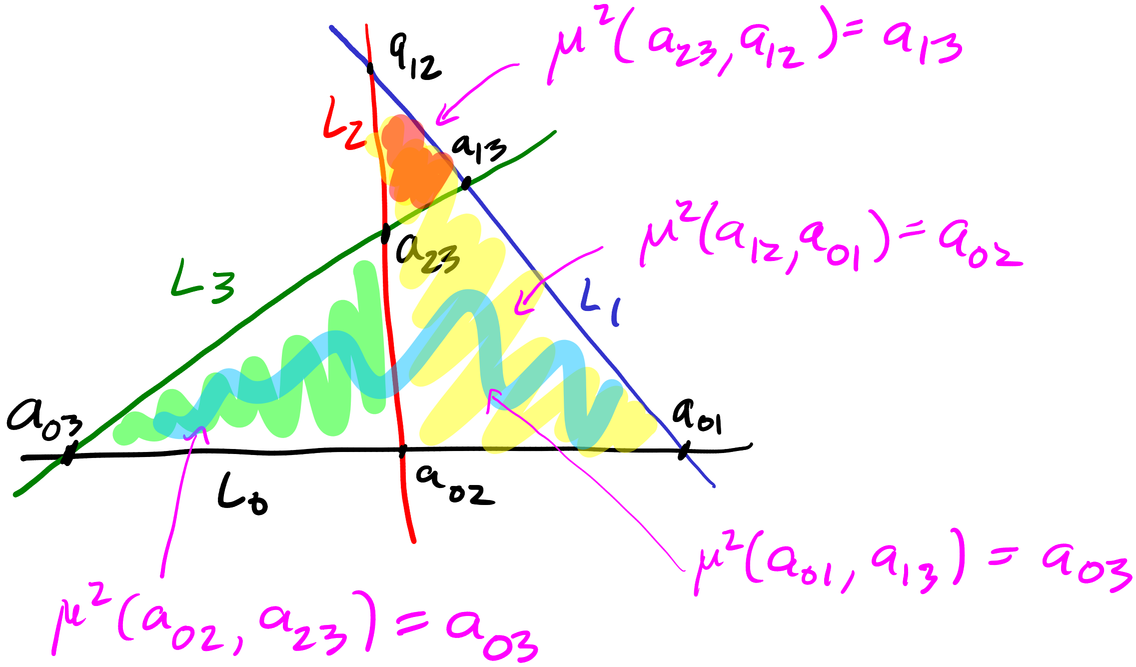


$i<j<k$

$$\mu^2 : \hom(L_1, L_2) \otimes \hom(L_0, L_1) \to \hom(L_0, L_2)$$



So $\mu^2(a_{12}, a_{01}) = a_{02}$



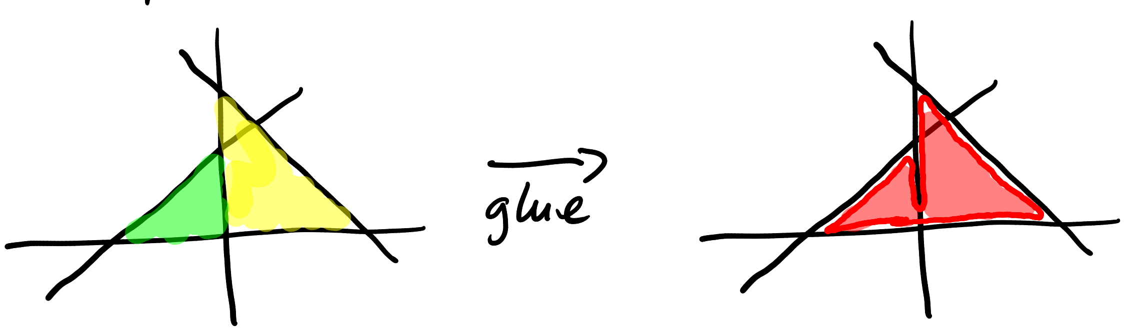
In summary $\mu^2(a_{jk}, a_{ij}) = a_{ik}$ for $i<j<k$

This operation is associative:

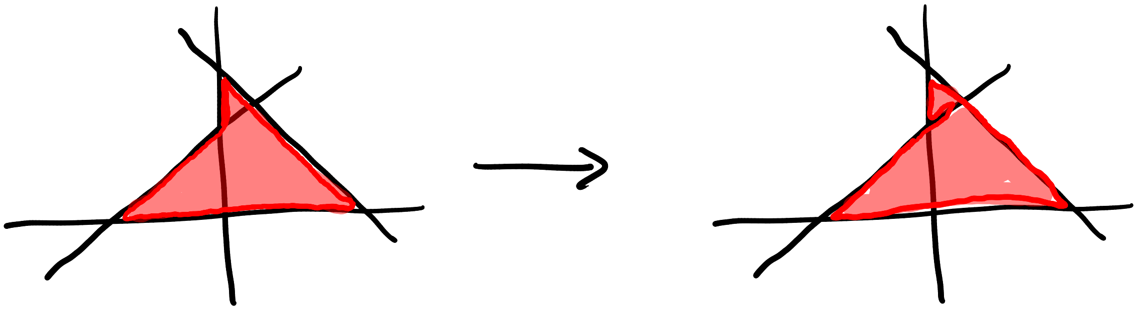
$$\mu^2(a_{23}, \mu^2(a_{12}, a_{01})) = \mu^2(a_{23}, a_{02}) = a_{03}$$

$$\mu^2(\mu^2(a_{23}, a_{12}), a_{01}) = \mu^2(a_{13}, a_{01}) = a_{03}$$

This associativity is "witnessed" by a 1-parameter family of 4-gons that interpolates between the two products

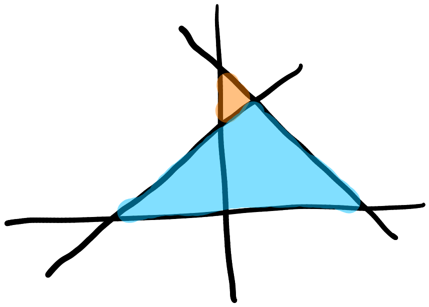


$\mu^2(a_{23}, \mu^2(a_{12}, a_{01}))$ 4-gon with slit on L_2



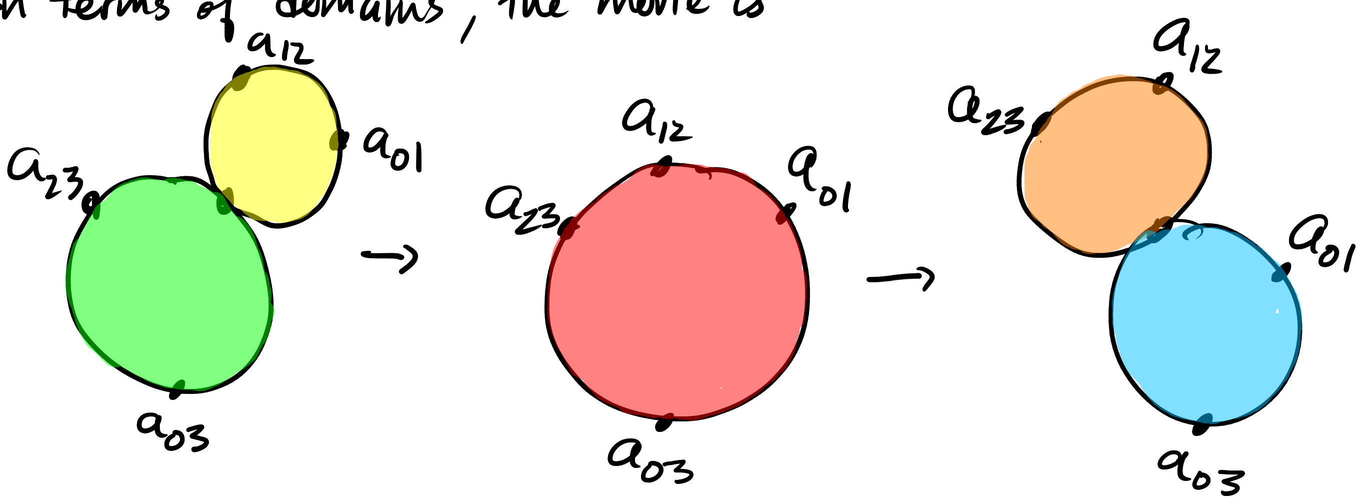
4-gon without slit

4-gon with slit on L_3

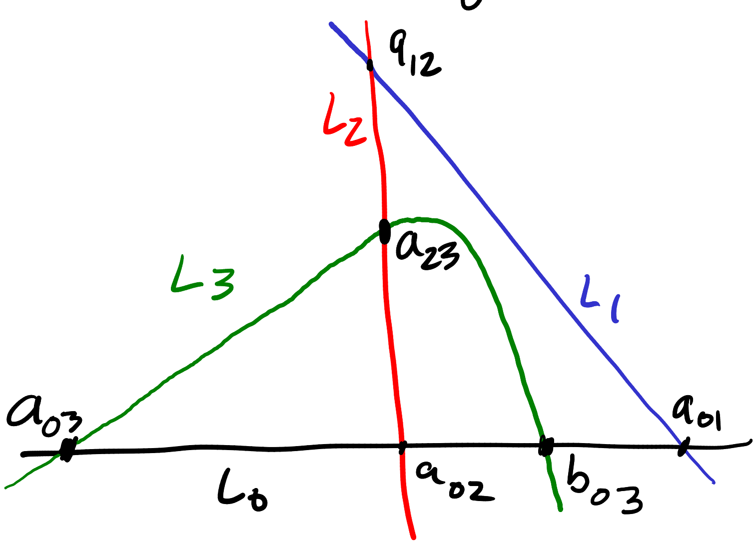


$$\mu^2(\mu^2(a_{23}, a_{12}), a_{01})$$

In terms of domains, the movie is



Next, let us change L_3 so that it is not straight



$\hom(L_0, L_1) = \langle a_{01} \rangle$

$\hom(L_0, L_2) = \langle a_{02} \rangle$

$\hom(L_0, L_3) = \langle a_{03}, b_{03} \rangle$

$\hom(L_1, L_2) = \langle a_{12} \rangle$

$\hom(L_1, L_3) = 0$

$\hom(L_2, L_3) = \langle a_{23} \rangle$.

We still have $\mu^2(a_{12}, a_{01}) = a_{02}$
$\mu^2(a_{23}, a_{02}) = a_{03}$

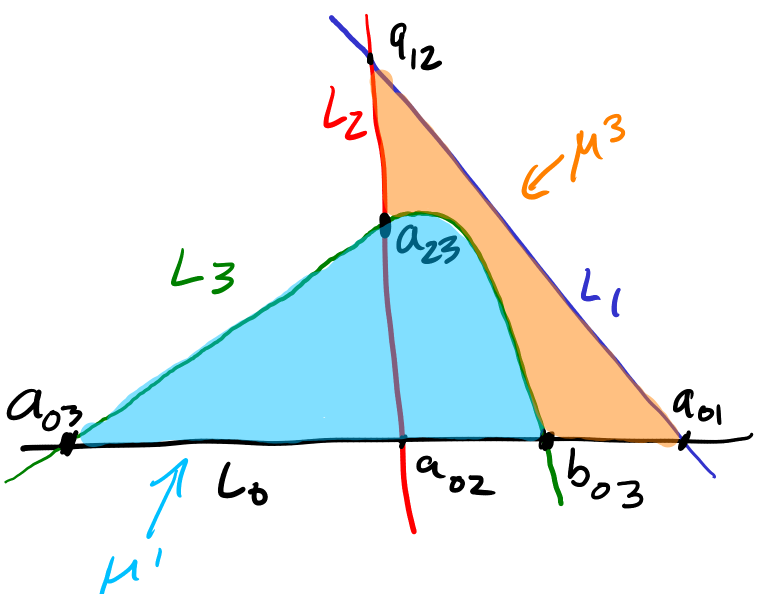
but $\mu^2(a_{23}, a_{12}) = 0$

So $\mu^2(a_{23}, \mu^2(a_{12}, a_{01})) = \mu^2(a_{23}, a_{02}) = a_{03}$

$\mu^2(\mu^2(a_{23}, a_{12}), a_{01}) = \mu^2(0, a_{01}) = 0$

And (strict) associativity fails!

But we also have μ^1 and μ^3 now!



$\mu^1(b_{03}) = a_{03}$

$\mu^3(a_{23}, a_{12}, a_{01}) = b_{03}$

Thus $\mu^2(a_{23}, \mu^2(a_{12}, a_{01})) - \mu^2(\mu^2(a_{23}, a_{12}), a_{01})$

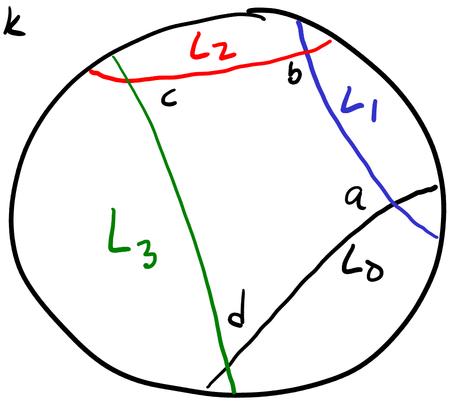
$= \quad a_{03} \quad - \quad 0$

$= \mu^1(\mu^3(a_{23}, a_{12}, a_{01}))$

So μ^2 is associative up to homotopy.

One last simple example

S = disk



$\hom(L_0, L_1) = \langle a \rangle$
$\hom(L_0, L_2) = 0$
$\hom(L_0, L_3) = \langle d \rangle$
$\hom(L_1, L_2) = \langle b \rangle$
$\hom(L_1, L_3) = 0$
$\hom(L_2, L_3) = \langle c \rangle$

Now $\mu^1 \equiv 0$ and $\mu^2 \equiv 0$ so μ^2 is associative.

But $\mu^3(c, b, a) = d \neq 0$.
So even though associativity holds, we still get some higher homotopical information.