The Stasheff associahedron.

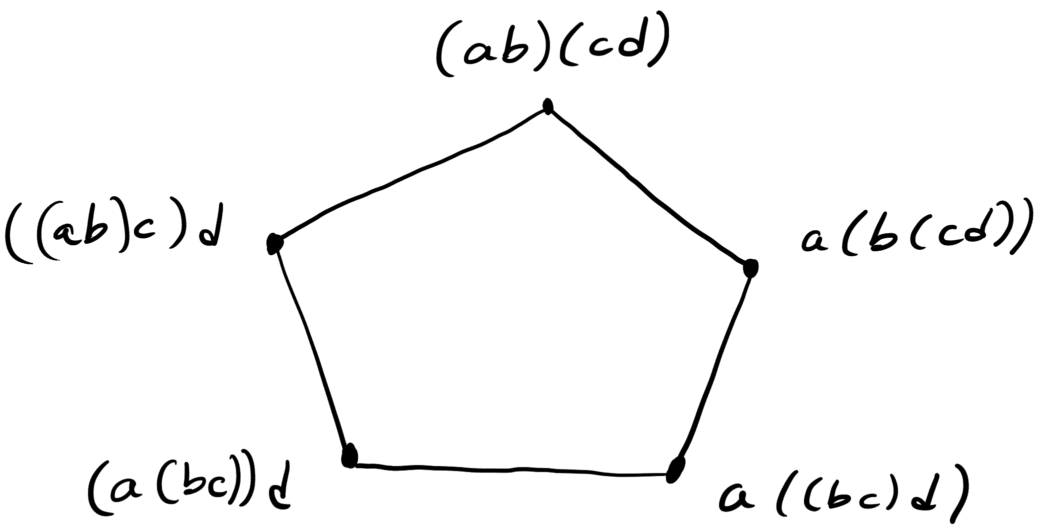
In basic algebra, we learn that if a binary operation $(a,b) \mapsto a\cdot b$ satisfies the trinary associative law $(a\cdot b)\cdot c = a\cdot(b\cdot c)$, then in fact there is a unique value for the iterated product

$$a_1 \cdot a_2 \cdot \cdots \cdot a_d \quad \text{for all } d > 0.$$

That is, we can insert the parentheses any way we wish and always get the same result.

Essentially, iterated application of the trinary assoc. law connects any two parenthesizations.

For 4 elements a,b,c,d this forms a pentagon



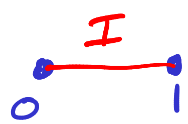
For homotopy coherent associativity (that is, A_∞-structures) it is not enough to know that any two parenthesizations may be connected, we need to remember "how" the were connected. Stasheff's Associahedron is a combinatorial object that indexes the data we need.

Associahedron as an abstract polytope.

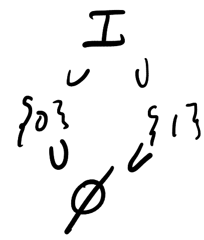
Any geometric polytope P (eg. dodecahedron) has a face poset $\{ F \mid F \text{ is a closed face of } P \}$

$$F_1 \leq F_2 \iff F_1 \subseteq F_2$$

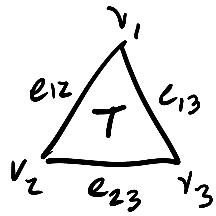
example: Interval

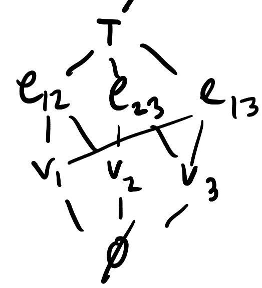


Poset



Triangle





An abstract polytope is a poset satisfying certain axioms that abstract some of the properties of the face poset of a geometric polytope.

The Stasheff Associahedron K_d is the (abstract) polytope whose nonempty faces correspond to partial parenthesizations of a string of d letters.

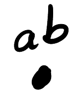
$F_1 \leq F_2 \iff F_2$ is obtained from F_1 by deleting sets of parentheses.

Note: "degenerate" parenthesizations such as (a)(b)(c) or (abc) are excluded here.

It turns out K_d may be realized geometrically; in fact we shall need a realization of it as a moduli space of Riemann surfaces.

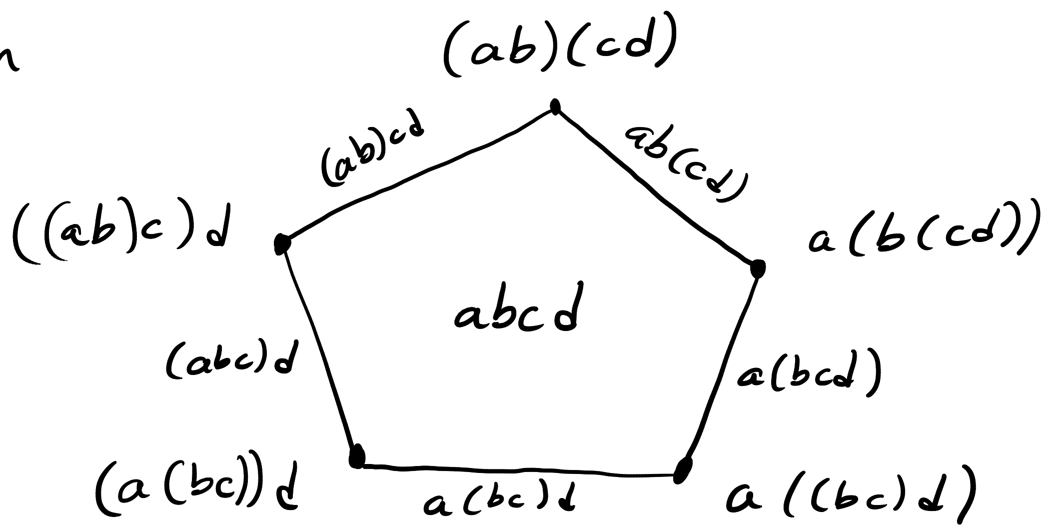
Since it takes $d-2$ sets of parens to fully parenthesize a d-fold product, $\dim K_d = d-2$.

K_2 = point =

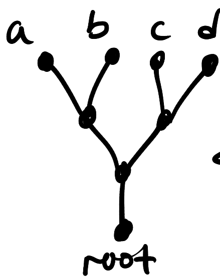


K_3 = interval =

K_4 = pentagon



Faces also correspond to planar d-leafed rooted trees



$\Longleftrightarrow$ (ab)(cd)

\# of internal edges = \# of sets of parens.

$F_1 \leq F_2 \iff F_2$ is obtained from F_1 by contracting internal edges.

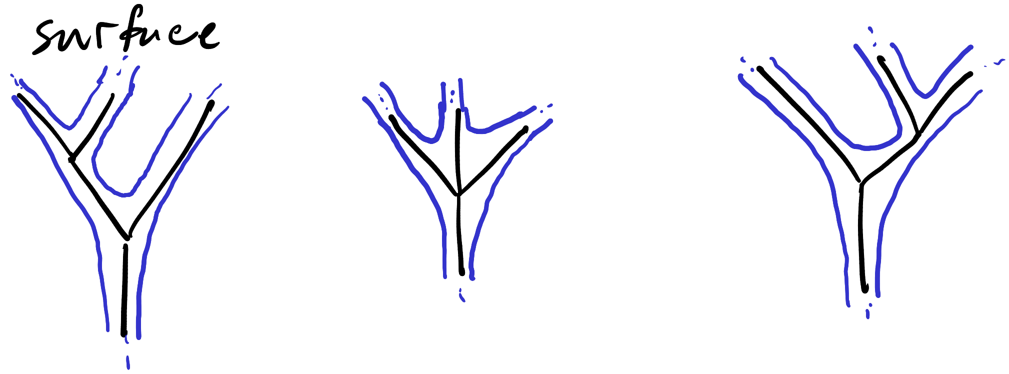
every internal vertex has valence ≥ 3
$\iff$ nondegenerate (not (a)bd)

Observe: the terms in the A_∞-associativity equations

$$0 = \sum (-1)^{\maltese} \mu^{d-m+1}(\underbrace{a \cdots a}_{d-m-n}\ \mu^m(\underbrace{a \cdots a}_{m}) \underbrace{a \cdots a}_{n})$$

Correspond to the codimension 1 faces of K_d
PLUS the degenerate cases $(a_d) \cdots a_1$, $a_d(a_{d-1}) \cdots a_1, \cdots$ $(a_d \cdots a_1)$
for the terms involving μ^1 and μ^d.

What is the connection to surfaces?
Recall that the trees are planar.
This means each tree can be thickened up to a surface



To make the connection precise we need to think about complex structures on the surfaces...