

A_∞ -categories and functors

- * Generalizes DG categories and functors.
- * More natural for Fukaya categories.
- * Also useful in DG category theory.

Towards A_∞ -algebras ($= A_\infty$ -category with one object)

DG algebra: $A = \bigoplus_{n \in \mathbb{Z}} A^n$ graded k -module
differential $d: A^n \rightarrow A^{n+1}$ $d^2 = 0$

$m: A \otimes A \rightarrow A$ multiplication, k -linear

Require m to be a chain map

$$d(m(a \otimes b)) = m(d(a \otimes b)) = m(d(a) \otimes b) + (-1)^{\deg(a)} m(a \otimes d(b))$$

$$d(ab) = (da) \cdot b + (-1)^{\deg a} a(db)$$

So m descends to a map $m: H^*(A) \otimes H^*(A) \rightarrow H^*(A)$

In a DG algebra, we require multiplication to be associative
 $m \circ (m \otimes 1) = m \circ (1 \otimes m)$

But the two sides are chain maps, so we could weaken this to the condition that they are
chain homotopic:

$$m \circ (m \otimes 1) \underset{\text{chain homotopy}}{\sim} m \circ (1 \otimes m)$$

This would suffice to guarantee that $m: H^*(A) \otimes H^*(A) \rightarrow H^*(A)$
is associative

We should really do this coherently:

- Specify a homotopy
- homotopies should be coherent for quadruple and n -fold products $(a_1 a_2 \cdots a_n)$

Def (A_∞ -algebra, with Seidel's sign convention)

An A_∞ -algebra consists of a graded k -module

$$A = \bigoplus_{n \in \mathbb{Z}} A^n$$

and maps

$$\mu^d: A^{\otimes d} \rightarrow A[2-d] \quad (\text{degree shift by } 2-d)$$

such that for all $d \geq 0$, $a_1, \dots, a_d \in A$

$$0 = \sum (-1)^{\star} \mu^{d-n+1}(a_1, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1)$$

where $\star = \deg(a_1) + \dots + \deg(a_n) - n$ and the sum is over all possible terms ($1 \leq m \leq d$, $0 \leq n \leq d-m$).

If $\mu^d = 0$ for all $d \geq 3$, then we get a DG algebra by setting $da = (-1)^{\deg(a)} \mu^1(a)$

$$m(a_2, a_1) = (-1)^{\deg(a_1)} \mu^1(a_2, a_1)$$

An A_∞ -algebra is called strictly unital if there is $e \in A^0$ such that $\mu^1(e) = 0$

$$\mu^2(a, e) = a = (-1)^{\deg(a)} \mu^2(e, a)$$

$$\text{for } d \geq 3 \quad \mu^d(\dots, e; \dots) = 0$$

An A_∞ -category is the "multi-object" version of this

Def An A_∞ -category A consists of

- a set of objects $Ob A$

- For each pair $X, Y \in Ob A$, a graded k -module $hom_A(X, Y)$

- For each sequence $X_0, X_1, \dots, X_d \in Ob A$, a map

$$\mu_A^d: hom_A(X_{d-1}, X_d) \otimes \dots \otimes hom_A(X_0, X_1) \longrightarrow hom_A(X_0, X_d)[2-d]$$

Such that the A_∞ -associativity equations hold for any d -tuple $(a_d, a_{d-1}, \dots, a_1)$ such that
 $\text{target}(a_i) = \text{source}(a_{i+1})$

A is called strictly unital if for each $X \in \text{Ob } A$ there is $e_X \in \overline{\text{hom}_A^0(X, X)}$ such that the unitality properties studied for A_∞ -algebras hold.

$\mu^1 \leftarrow$ differential

$\mu^2 \leftarrow$ product

$\mu^3 \leftarrow$ witness that μ^2 is associative up to homotopy

$\mu^4 \leftarrow$ coherence of associators for 4-tuple

A_∞ -functors preserve composition of morphisms up to homotopy

Dif An A_∞ -functor $F: A \rightarrow B$ between A_∞ -categories consists of: A map $F: \text{Ob } A \rightarrow \text{Ob } B$ on objects for each $d \geq 1$ and tuple $X_0, X_1, \dots, X_d \in \text{Ob } A$ a map

$$F^d: \text{hom}_A(X_{d-1}, X_d) \otimes \cdots \otimes \text{hom}_A(X_0, X_1) \rightarrow \text{hom}_B(FX_0, FX_d)[1-d]$$

such that

$$\begin{aligned} & \sum_r \sum_{s_1, \dots, s_r} \mu_B^r(F^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, F^{s_1}(a_{s_1}, \dots, a_1)) \\ &= \sum_{m,n} (-1)^{\frac{d(d-m+1)}{2}} F^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu_A^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) \end{aligned}$$

$F^1: \text{hom}_A(X, Y) \rightarrow \text{hom}_B(FX, FY)$ chain map

$F^2: \text{hom}_A(Y, Z) \otimes \text{hom}_A(X, Y) \rightarrow \text{hom}_B(FX, FZ)[-1]$

is a witness that $F^1(\mu_A^2(f, g)) \sim \mu_B^2(F^1(f), F^1(g))$
and so on.

F is strictly unital if $F^1(e_X) = e_{FX}$ and
 $F^d(\dots, e_X, \dots) = 0$ for any $d \geq 2$.

A_∞ -functors may be composed

$$(G \circ F)^d(a_d, \dots, a_1)$$

$$= \sum_r \sum_{s_1 \dots s_r} G^r(F^{s_1}(a_1, \dots, a_{s_1}), \dots, F^{s_r}(a_s, \dots, a_1))$$

This composition is strictly associative and strictly unital

The collection of A_∞ -functors $\text{Fun}_{A_\infty}(A, B)$
is itself an A_∞ -category

$\text{hom}_{\text{Fun}_{A_\infty}(A, B)}^P(F, G)$ consists of sequences of multilinear

maps (T^0, T^1, \dots) $T^0 \in \text{hom}_B^P(FX, GX)$

$T^d: \text{hom}_A(X_{d-1}, X_d) \otimes \dots \otimes \text{hom}_A(X_0, X_1) \rightarrow \text{hom}_B(FX_0, GX_d)[P-d]$

There are A_∞ operations that I will not write explicitly now.

A_∞ functors are still useful if we are interested
in DG categories (= A_∞ -categories with $\mu^d = 0, d \geq 3$) 5

Fact 1: If A is an A_∞ category, and B is a DG category, $\text{Fun}_{A_\infty}(A, B)$ is a DG category

Fact 2: (Kontsevich, Fuente)

Under some mild conditions (e.g. if k is a field)
then for DG categories A and B ,

$\text{Fun}_{A_\infty}(A, B)$ is an internal hom object

for the homotopy category of DG categories localized
at the quasi-equivalences

$$\text{Ho}(\text{DG cat}_k) = \text{DG cat}_k [W_{\text{quasi-equiv}}^{-1}]$$

Whereas $\text{Fun}_{\text{DG}}(A, B)$ does not enjoy this property.

This means that $\text{Fun}_{A_\infty}(A, B)$ is the "homotopically correct"
DG category of maps $A \rightarrow B$.

Pictures for the equations in terms of planar trees

A_∞ -equations:

cations:

$$\sum \begin{matrix} a \\ a \\ a \end{matrix} + \begin{matrix} a \\ a \\ a \\ a \end{matrix} = 0$$

A₀₀-functor equations

Function composition

A hand-drawn diagram showing a central vertical line from which several branches extend. The top-most branch has three smaller branches labeled 'F' at their tips. Below this cluster is a label 'T'. To the left of the main stem, there is a large, open, curved bracket-like shape labeled 'S'.

μ' of a natural transformation T :

μ^d of T :

A hand-drawn diagram illustrating a branching process. On the left, there is a large, open bracket-like symbol with a wavy top and a straight bottom, containing the letter Σ . To its right, a vertical line descends from a point labeled M_B . From this point, several branches extend upwards and to the right. The first branch on the left is labeled F below it. The second branch is labeled T above it. The third branch is labeled F below it. The fourth branch is labeled T above it. The fifth branch on the far right is partially cut off by the edge of the page.

T is used d times.
 If $\mu_B^d = 0$ for $d \geq 3$,
 then all trees of this
 shape yield zero.