A_∞-categories and functors

* Generalizes DG categories and functors.
* More natural for Fukaya categories.
* Also useful in DG category theory.

Towards A_∞-algebras ($=A_\infty$-category with one object)

<u>DG algebra</u>: $A = \bigoplus_{n\in\mathbb{Z}} A^n$ graded k-module

differential $d: A^n \to A^{n+1}$ $\quad d^2=0$

$m: A\otimes A \to A$ multiplication, k-linear

Require m to be a chain map

$$d(m(a\otimes b)) = m(d(a\otimes b)) = m(d(a)\otimes b) + (-1)^{\deg(a)} m(a\otimes d(b))$$

$$d(a\cdot b) = (da)\cdot b + (-1)^{\deg a} a\,(db)$$

So m descends to a map $m: H^\cdot(A)\otimes H^\cdot(A) \to H^\cdot(A)$

In a DG algebra, we require multiplication to be associative

$$m\circ(m\otimes 1) = m\circ(1\otimes m)$$

But the two sides are chain maps, so we could weaken this to the condition that they are <u>chain homotopic</u>:

$$m\circ(m\otimes 1) \underset{\substack{\text{chain}\\ \text{homotopy}}}{\sim} m\circ(1\otimes m)$$

This would suffice to gaurantee that $m: H^\cdot(A)\otimes H^\cdot(A)\to H^\cdot(A)$ is associative

We should really do this coherently:

- specify a homotopy
- homotopies should be coherent for quadruple and n-fold products $(a_1 a_2 \cdots a_n)$

<u>Def</u> (A_∞-algebra, with Seidel's sign convention)

An <u>A_∞-algebra</u> consists of a graded k-module
$$A = \bigoplus_{n\in\mathbb{Z}} A^n$$
and maps
$$\mu^d : A^{\otimes d} \longrightarrow A[2-d] \quad \text{(degree shift by } 2-d)$$
such that for all $d>0$, $a_1,\dots,a_d \in A$
$$0 = \sum (-1)^{\maltese} \mu^{d-m+1}(a_d,\dots,a_{n+m+1},\mu^m(a_{n+m},\dots,a_{n+1}),a_n,\dots,a_1)$$

where $\maltese = \deg(a_1)+\dots+\deg(a_n)-n$ and the sum is over all possible terms ($1\le m\le d$, $0\le n\le d-m$).

If $\mu^d \equiv 0$ for all $d\geq 3$, then we get a DG algebra by setting $da = (-1)^{\deg(a)}\mu^1(a)$
$$m(a_2,a_1) = (-1)^{\deg(a_1)}\mu^1(a_2,a_1)$$

An A_∞-algebra is called <u>strictly unital</u> if there is $e\in A^0$ such that $\mu^1(e)=0$
$$\mu^2(a,e) = a = (-1)^{\deg(a)}\mu^2(e,a)$$
for $d\geq 3$ $\mu^d(\dots,e,\dots)=0$

An A_∞-category is the "multi-object" version of this

<u>Def</u> An A_∞-category $\mathcal{A}$ consists of

- a set of objects $Ob\,\mathcal{A}$
- For each pair $X,Y\in Ob\,\mathcal{A}$, a graded k-module $hom_{\mathcal{A}}(X,Y)$
- For each sequence $X_0,X_1,\dots,X_d\in Ob\,\mathcal{A}$, a map

$$\mu^d_{\mathcal{A}} : hom_{\mathcal{A}}(X_{d-1},X_d)\otimes\cdots\otimes hom_{\mathcal{A}}(X_0,X_1) \longrightarrow hom_{\mathcal{A}}(X_0,X_d)[2-d]$$

Such that the A_∞-associativity equations hold for any d-tuple $(a_d, a_{d-1}, \dots, a_1)$ such that
$\mathrm{target}(a_i) = \mathrm{source}(a_{i+1})$

$\mathcal{A}$ is called <u>strictly unital</u> if for each $X \in \mathrm{Ob}\,\mathcal{A}$ there is $e_X \in \hom^0_{\mathcal{A}}(X,X)$ such that the unitality properties stated for A_∞-algebras hold.

$\mu^1 \leftarrow$ differential
$\mu^2 \leftarrow$ product
$\mu^3 \leftarrow$ witness that μ^2 is associative up to homotopy
$\mu^4 \leftarrow$ coherence of associators for 4-tuple
$\vdots$

A_∞-functors preserve compostion of morphisms up to homotopy

Def An <u>A_∞-functor</u> $F : \mathcal{A} \to \mathcal{B}$ between A_∞-categories consists of: A map $F : \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$ on objects for each $d \geq 1$ and tuple $X_0, X_1, \dots, X_d \in \mathrm{Ob}\,\mathcal{A}$ a map

$$F^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{B}}(FX_0, FX_d)[1-d]$$

Such that

$$\sum_r \sum_{s_1, \dots, s_r} \mu^r_{\mathcal{B}}(F^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, F^{s_1}(a_{s_1}, \dots, a_1))$$

$$= \sum_{m,n} (-1)^{\maltese} F^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$F^1 : \hom_{\mathcal{A}}(X,Y) \to \hom_{\mathcal{B}}(FX,FY)$ chain map

$F^2 : \hom_{\mathcal{A}}(Y,Z) \otimes \hom_{\mathcal{A}}(X,Y) \to \hom_{\mathcal{B}}(FX,FZ)[-1]$

is a witness that $F^1(\mu^2_{\mathcal{A}}(f,g)) \sim \mu^2_{\mathcal{B}}(F^1(f),F^1(g))$
and so on.

F is <u>strictly unital</u> if $F^1(e_X) = e_{FX}$ and
$F^d(\cdots, e_{X_i}, \cdots) = 0$ for any $d \geq 2$.

A_∞-functors may be composed

$$(G \circ F)^d(a_d, \dots, a_1)$$
$$= \sum_r \sum_{s_1 \cdots s_r} G^r(F^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, F^{s_1}(a_{s_1} \cdots a_1))$$

This composition is strictly associative and strictly unital

The collection of A_∞-functors $\mathrm{Fun}_{A_\infty}(\mathcal{A},\mathcal{B})$
is itself an A_∞-category

$\hom^p_{\mathrm{Fun}_{A_\infty}(\mathcal{A},\mathcal{B})}(F,G)$ consists of sequences of multilinear

maps $(T^0, T^1, \dots)$ $\quad T^0 \in \hom^p_{\mathcal{B}}(FX,GX)$

$$T^d : \hom_{\mathcal{A}}(X_{d-1},X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0,X_1) \longrightarrow \hom_{\mathcal{B}}(FX_0,GX_d)[p-d]$$

There are A_∞ operations that I will not write explicitly
now.

A_∞ functors are still useful if we are interested in DG categories ($=$ A_∞-categories with $\mu^d \equiv 0$, $d \geq 3$)

Fact 1: If $\mathcal{A}$ is an A_∞ category, and $\mathcal{B}$ is a DG category, $\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{B})$ is a DG category

Fact 2: (Kontsevich, Faonte)
Under some mild conditions (e.g. if k is a field) then for DG categories $\mathcal{A}$ and $\mathcal{B}$,

$\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{B})$ is an internal hom object

for the homotopy category of DG categories localized at the quasi-equivalences

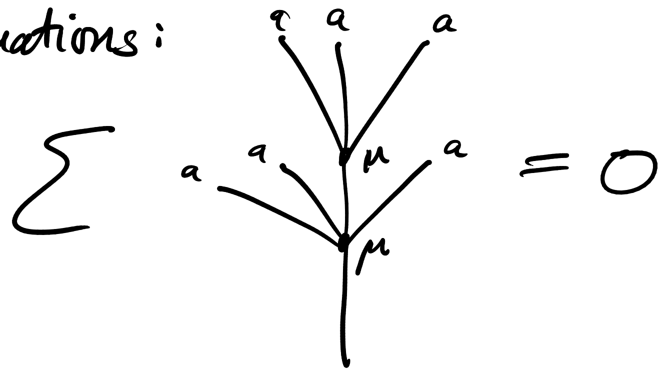
$$\mathrm{Ho}(\mathrm{DGcat}_k) = \mathrm{DGcat}_k[W^{-1}_{\text{quasi-equiv}}]$$

Whereas $\mathrm{Fun}_{DG}(\mathcal{A}, \mathcal{B})$ does not enjoy this property.

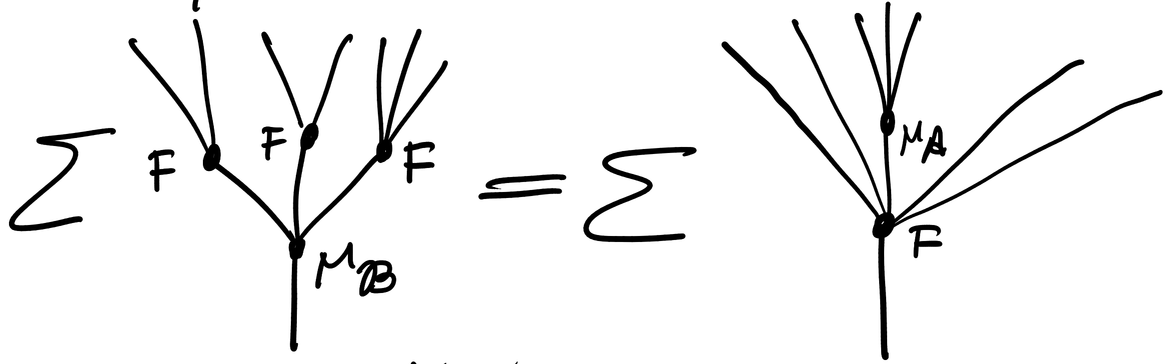
This means that $\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{B})$ is the "homotopically correct" DG category of maps $\mathcal{A} \to \mathcal{B}$.

Pictures for the equations in terms of planar trees

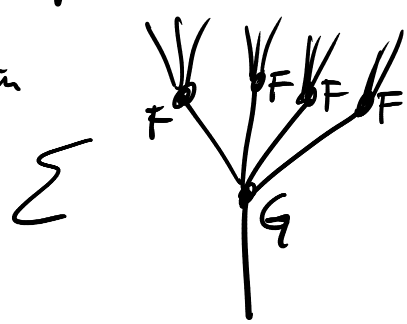
A_∞ - equations:



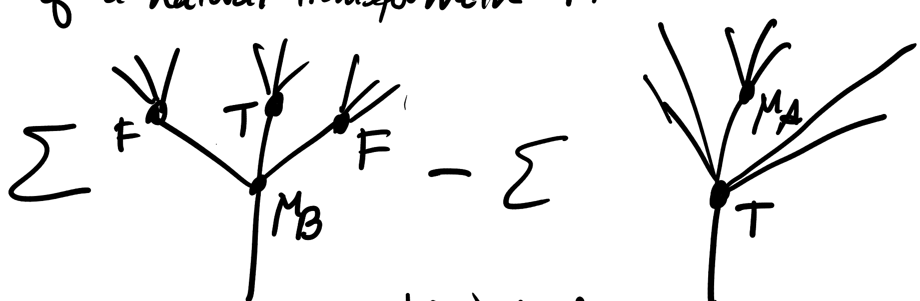
A_∞ - functor equations



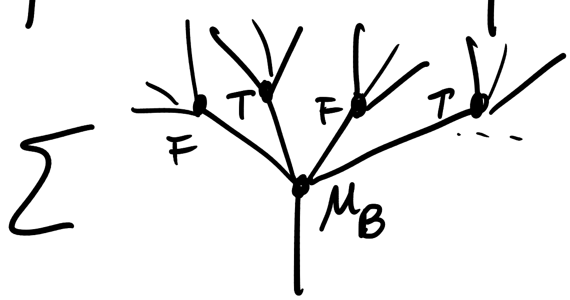
Functor composition



μ^1 of a natural transformation T:



μ^d of T:



T is used d times.
If $\mu_B^d = 0$ for $d \geq 3$, then all trees of this shape yield zero.