Math 417 Review Sheet: Homomorphisms, Cosets, and Quotient Groups

We use capital letters $G, H, \dots$ for groups, and Greek letters $\phi, \pi, \dots$ for group homomorphisms.

Definition 1 (Homomorphism)**.** Let G and H be groups. A function $\phi : G \to H$ is called a *homomorphism* if

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \text{ for all } g_1, g_2 \in G.$$

(It should be understood that the product on the left-hand side is the group operation in G, and the product on the right-hand side is the group operation in H).

Theorem 2 (Basic facts about homomorphisms)**.** *Let $\phi : G \to H$ be a homomorphism. Then*

(1) *ϕ takes the identity to the identity: $\phi(e_G) = e_H$.*
(2) *ϕ takes inverses to inverses: $\phi(g^{-1}) = \phi(g)^{-1}$.*
(3) *If A is a subgroup of G, then $\phi(A)$ is subgroup of H.*
(4) *If B is a subgroup of H, then $\phi^{-1}(B)$ is a subgroup of G.*

Definition 3 (Normal subgroup)**.** A subgroup N in a group G is called *normal* if is closed under conjugation by elements of G:

$$gng^{-1} \in N \text{ for all } n \in N \text{ and } g \in G.$$

Definition 4 (Kernel)**.** Let $\phi : G \to H$ be a homomorphism. The *kernel* of ϕ is the set

$$\ker(\phi) = \phi^{-1}(e_H) = \{g \in G \mid \phi(g) = e_H\}.$$

Theorem 5. *The kernel of a homomorphism is a normal subgroup of the domain; that is, if $\phi : G \to H$ is a homomorphism, then $\ker(\phi)$ is a normal subgroup of G.*

Theorem 6. *A homomorphism $\phi : G \to H$ is injective if and only if $\ker(\phi) = \{e_G\}$.*

Definition 7 (Cosets)**.** Let H be a subgroup of a group G. A *left coset* of H in G is a set of the form

$$aH = \{ah \mid h \in H\}.$$

The set of all left cosets of H in G is denoted

$$G/H = \{aH \mid a \in G\}.$$

The *index* of H in G is the cardinality of G/H. It is denoted $[G : H]$.

Theorem 8 (Structural facts upon which Lagrange's theorem is based)**.** *Let H be a subgroup of G.*

(1) *The left cosets of H in G form a partition of G; that is, the cosets are nonempty, pairwise disjoint, and their union is all of G.*
(2) *The corresponding equivalence relation, denoted $\sim_H$, is given by*

$$a \sim_H b \iff b^{-1}a \in H.$$

(3) *All left cosets have the same cardinality; more specifically, the operation of left multiplication by ba^{-1} is a bijective function $aH \to bH$. Since $H = eH$ is itself a left coset, all left cosets of H have the same cardinality as H.*

Theorem 9 (Lagrange's theorem)**.** *Let G be a finite group, and let H be a subgroup of G. Then the order of H divides the order of G, and the index $[G : H]$ is the ratio of these orders:*

$$[G : H] = \frac{|G|}{|H|}.$$

Even if G or H is not finite, the equation

$$|G| = |H| \cdot [G : H]$$

is valid, it being understood that $\infty \cdot (\text{finite}) = \infty$, and $\infty \cdot \infty = \infty$.

Theorem 10 (Existence of the quotient group)**.** *Let N be a normal subgroup of G. Let $\pi : G \to G/N$ be the function that takes an element of G to its left coset: $\pi(a) = aN$. Then*

(1) *There is a unique binary operation on G/N that makes π into a homomorphism.*
(2) *This operation is given by the formula*

$$(aN)(bN) = (ab)N.$$

Using this operation as the group operation on G/N, the following facts are true:

(1) *The identity element for G/N is $N = eN$.*
(2) *The inverses in G/N are given by $(aN)^{-1} = (a^{-1})N$.*
(3) *The function $\pi : G \to G/N$ is a surjective homomorphism whose kernel is N.*

Remark 11. If H is a subgroup of G that is **not normal**, then the construction of the quotient group **does not work**: there does not exist a well-defined binary operation on G/H such that $\pi : G \to G/H$ is a homomorphism.