Math 417 Review Sheet: Isomorphism Theorems

We use capital letters $G, H, \dots$ for groups, and Greek letters $\phi, \pi, \dots$ for group homomorphisms.

Theorem 1 (First isomorphism theorem)**.** *Let $\phi : G \to H$ be a surjective homomorphism. Then $G/\ker(\phi)$ is isomorphic to H.*

Theorem 2 (First isomorphism theorem with a specific isomorphism)**.** *Let $\phi : G \to H$ be a surjective homomorphism. Let $N = \ker(\phi)$. Then the function $\tilde{\phi} : G/N \to H$ defined by $\tilde{\phi}(aN) = \phi(a)$ is well-defined and $\tilde{\phi}$ is an isomorphism.*

Theorem 3 (First isomorphism theorem without the surjectivity hypothesis)**.** *Let $\phi : G \to H$ be a homomorphism. Then $G/\ker(\phi)$ is isomorphic to $\phi(G)$.*

Theorem 4 (First isomorphism theorem as a slogan)**.** *The image of a homomorphism is isomorphic to the domain modulo the kernel.*

Remark 5. The first isomorphism theorem is also called the "homomorphism theorem" (Goodman). It also deserves to be called the "fundamental theorem of homomorphisms." It is similar to the fundamental theorem of calculus in that it is the fundamental theorem for this area of mathematics.

Theorem 6 (Correspondence theorem)**.** *Let $\phi : G \to H$ be a surjective homomorphism.*

(1) *If B is a subgroup of H, then $\phi^{-1}(B)$ is a subgroup of G containing $\ker(\phi)$.*
(2) *Conversely, if A is a subgroup of G containing $\ker(\phi)$, then there is a unique subgroup B of H such that $A = \phi^{-1}(B)$.*
(3) *Thus there is a bijective correspondence*

$$\{\textit{subgroups of } H\} \leftrightarrow \{\textit{subgroups of } G \textit{ containing } \ker(\phi)\}.$$

(4) *Moreover, normal subgroups correspond to normal subgroups: there is a bijective correspondence*

$$\{\textit{normal subgroups of } H\} \leftrightarrow \{\textit{normal subgroups of } G \textit{ containing } \ker(\phi)\}.$$

Theorem 7 (Correspondence theorem in the case of a quotient)**.** *Let G be a group, let N be a normal subgroup of G, and let $\pi : G \to G/N$ be the quotient homomorphism.*

(1) *If B is a subgroup of G/N, then $\pi^{-1}(B)$ is a subgroup of G containing N.*
(2) *Conversely, if A is a subgroup of G containing N, then there is a unique subgroup B of G/N such that $A = \pi^{-1}(B)$; in this situation, $B = A/N$.*
(3) *Thus there is a bijective correspondence*

$$\{\textit{subgroups of } G/N\} \leftrightarrow \{\textit{subgroups of } G \textit{ containing } N\}.$$

(4) *Moreover, normal subgroups correspond to normal subgroups: there is a bijective correspondence*

$$\{\textit{normal subgroups of } G/N\} \leftrightarrow \{\textit{normal subgroups of } G \textit{ containing } N\}.$$

Theorem 8 (Quotients by corresponding normal subgroups are isomorphic)**.** *Let $\phi : G \to H$ be a surjective homomorphism. Suppose that B is a normal subgroup of H corresponding to $A = \phi^{-1}(B)$. Then G/A is isomorphic to H/B.*
More specifically, the function $\psi : G/A \to H/B$ defined by $\psi(gA) = \phi(g)B$ is an isomorphism.

Theorem 9 (Quotient of a quotient)**.** *Let N and K be normal subgroups of G, with $N \subseteq K$. Then G/K is isomorphic to $(G/N)/(K/N)$.*

Remark 10. (1) Theorem 9 is a special case of Theorem 8, where we apply Theorem 8 to the case where $H = G/N$, ϕ is the quotient homomorphism $\pi : G \to G/N$, $A = K$, and $B = K/N$.

(2) These theorems are often called the **third isomorphism theorem**. (Note that in the lecture I called them the second isomorphism theorem, since indeed they come second in my presentation.)

Theorem 11 (Factorization theorem)**.** *Let $\phi : G \to H$ be a homomorphism, and let N be a normal subgroup of G such that $N \subseteq \ker(\phi)$. Then the function $\tilde{\phi} : G/N \to H$ given by $\tilde{\phi}(aN) = \phi(a)$ is a well-defined homomorphism.*

Theorem 12 (Diamond isomorphism theorem)**.** *Let $\phi : G \to H$ be a homomorphism, and let $N = \ker(\phi)$. Let A be a subgroup of G. Then*

(1) $\phi^{-1}(\phi(A)) = AN$, *and*
(2) $AN/N \cong \phi(A) \cong A/(A \cap N)$.

Theorem 13 (Diamond isomorphism theorem as a statement about subgroups)**.** *Let G be a group, let N be a normal subgroup of G, and let A be a subgroup of G (not assumed to be normal). Then AN is a subgroup of G, N is a normal subgroup of AN, $A \cap N$ is a normal subgroup of A, and there is an isomorphism*

$$AN/N \cong A/(A \cap N).$$

Remark 14. (1) One way to think about the diamond isomorphism theorem is that it is complementary to the correspondence theorem. The correspondence theorem tells us that a subgroup A of G that contains N corresponds to a subgroup of G/N, while the diamond isomorphism theorem tells us what happens when A is a subgroup of G that does not necessarily contain N.
(2) Notice that, if we assumed that $N \subseteq A$ in the diamond isomorphism theorem, we would then have $AN = A$ and $A \cap N = N$. Thus the theorem would tell us that $A/N \cong A/N$, which is trivial.
(3) The diamond isomorphism theorem is often called the **second isomorphism theorem**. (Note that in the lecture I called it the third isomorphism theorem, since indeed it comes third in my presentation.)