Math 417: Final Exam Practice problems

1. Let G be the set of 3-by-3 matrices with the property that there is exactly one nonzero entry in each row, exactly one nonzero entry in each column, and the nonzero entries are always $+1$ or -1. Prove that G is isomorphic to the semidirect product of S_3 and H, where H is the group of 3-by-3 matrices that are diagonal with ± 1 along the diagonal.

2. Suppose that $G \cong \mathbb{Z}_5 \rtimes_\alpha \mathbb{Z}_3$ is a semidirect product of $\mathbb{Z}_5$ and $\mathbb{Z}_3$ with respect to a homomorphism $\alpha : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_5)$. Show that α is trivial and that $G \cong \mathbb{Z}_5 \times \mathbb{Z}_3$. Is G a cyclic group?

3. Consider the vector space $\mathbb{R}^n$. Let $G = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ be the group of nonzero real numbers with multiplication. Show that the multiplication of vectors by scalars
$$G \times \mathbb{R}^n \to \mathbb{R}^n,\ (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v}$$
defines an action of G on $\mathbb{R}^n$.

4. Consider the group D_4, the symmetries of a square. Let V be the set of vertices of the square, and let E be the set of edges of the square. Go through each of the 8 elements of D_4 and answer the questions: How many elements of V does it fix? How many elements of E does it fix?

5. Let a group G act on a set X. Let $Y \subseteq X$ be a subset, and define
$$G_Y = \{g \in G \mid \forall y \in Y,\ g \cdot y = y\}$$
to be the set of group elements that fix every element of Y. Show that G_Y is a subgroup of G.

6. Let a group G act on itself by conjugation. Show from the definitions that the kernel of this action equals the center of G.

7. Find the number of orbits in $\{1,2,3,4,5,6,7,8\}$ under the action of the subgroup of S_8 generated by (13) and (247).

8. How many ways are there to divide a set of 10 people into two sets of 5?

9. How many ways are there to seat 7 people around a round table, if we regard two arrangements that differ by a rotation as the same?

10. How many ways are there to color the edges of a square with 4 colors (if we regard colorings that differ by the action of an element of D_4 as being the same)?

11. Write out the conjugacy classes in S_4. Write out the class equation for S_4.

12. Let G be a finite group, and let p be a prime number dividing $|G|$. Let P be a subgroup of G whose order is a power of p, and which is normal. Show that any p-Sylow subgroup of G must contain P.

13. Show that every group of order 45 has a normal subgroup of order 9.