Math 417: Exam 2 Practice problems

Note: The problems on the exam may be similar in spirit to the ones given here, or they may be somewhat different. The point of this practice sheet is to develop a familiarity with the underlying theory and concepts.

1. Suppose that $\varphi : \mathbb{Z} \to \mathbb{Z}_7$ is a homomorphism, and that $\varphi(1) = [4]$. Find $\varphi(25)$ and $\ker(\varphi)$.

2. Find the subgroup of S_4 generated by the set $\{(12), (34)\}$.

3. Show that if $\varphi : G \to G'$ is a homomorphism and $|G|$ is prime, then φ is either injective or trivial. (For any groups G and G', the trivial homomorphism is the function defined by $\varphi(x) = e$ for all $x \in G$.)

4. For $g \in G$, we have the function $L_g : G \to G$, $L_g(x) = gx$. Find an example where L_g is a homomorphism, and find an example where L_g is not a homomorphism.

5. For $g \in G$, show that the function $C_g : G \to G$, $C_g(x) = gxg^{-1}$ is an isomorphism.

6. How many elements does the quotient group $\mathbb{Z}_{10}/\langle[4]\rangle$ have?

7. Let N be a normal subgroup of G, and let $m = [G : N]$ be the number of cosets of N in G. Show that for every $g \in G$, $g^m \in N$.

8. Let $\zeta_n = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$. Let $G = \langle\zeta_n\rangle \leq \mathbb{C}^*$ be the subgroup of group of nonzero complex numbers (with multiplication) generated by ζ_n. Show that G is isomorphic to $\mathbb{Z}_n$.

9. Find a group G and two normal subgroups $H \lhd G$, $K \lhd G$ such that H is isomorphic to K but G/H is not isomorphic to G/K.

10. Find a complete list of all subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

11. Let $\varphi : G \to \overline{G}$ be a group homomorphism. Let $N \lhd \overline{G}$ be a normal subgroup. Show that $\varphi^{-1}(N)$ is a normal subgroup of G.

12. (This is too long for an exam problem, and a similar problem appeared on the homework.) In the group $G = \mathbb{Z}_{24}$, let $H = \langle[4]\rangle$, and $K = \langle[8]\rangle$. Note $K \leq H \leq G$.
    (a) List the cosets in G/H, showing the elements in each coset.
    (b) List the cosets in G/K, showing the elements in each coset.
    (c) List the cosets in H/K, showing the elements in each coset.
    (d) List the cosets in $(G/H)/(H/K)$, showing the elements in each coset.
    (e) Write down the correspondence between G/H and $(G/K)/(H/K)$ that is gauranteed by Proposition 2.7.14.

13. Let $G = \mathbb{Z}_{512} \times \mathbb{Z}_{1729}$, and let H be the subgroup generated by $([0], [1])$. Show that G/H is isomorphic to $\mathbb{Z}_{512}$. (It's better not to try to list all 885248 elements of G ☺).

14. Let G be the group consisting of the four matrices

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

    Show that G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.