Lecture 19 The orbit-counting theorem
(Burnside or Cauchy-Frobenius lemma)

Last time: <u>orbit-stabilizer theorem</u>. If a group G acts on a set X, and $x \in X$, there is a bijection.

$$\psi : G/\mathrm{Stab}(x) \to G\cdot x$$

In particular, if X and G are finite sets, we have an equation.

$$|G\cdot x| = |G| / |\mathrm{Stab}(x)|$$

Here is another question: Assume X and G are finite.
<u>How many</u> orbits are there? There is a rather nice formula for this that follows from the orbit-stabilizer theorem together with a bit of clever arithmetic.

First: recall $\mathrm{Stab}(x) = \{ g \in G \mid g\cdot x = x \} \subseteq G$
Also define $\mathrm{Fix}(g) = \{ x \in X \mid g\cdot x = x \} \subseteq X$
$\mathrm{Fix}(g)$ is the set of <u>fixed points</u> of g acting in X.

<u>Orbit-counting theorem</u> Assume G and X are finite
Then

$$\#\text{ of orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

"The number of orbits equals the average number of fixed points of an element of G"

Proof: consider the set $\Gamma = \{(g,x) \mid g\cdot x = x\} \subseteq G \times X$ of pairs of a group element and a point fixed by it.

We count Γ is two ways

$$|\Gamma| = \sum_{x\in X}(\#\text{ of } g \text{ such that } (g,x)\in\Gamma) = \sum_{x\in X} |\mathrm{Stab}(x)|$$

$$|\Gamma| = \sum_{g\in G}(\#\text{ of } x \text{ such that } (g,x)\in\Gamma) = \sum_{g\in G} |\mathrm{Fix}(g)|$$

So $\displaystyle\sum_{x\in X} |\mathrm{Stab}(x)| = \sum_{g\in G} |\mathrm{Fix}(g)|$

Now $|\mathrm{Stab}(x)| = |G|/|G\cdot x|$ by orbit-stabilizer theorem,

so $\displaystyle\sum_{x\in X} \frac{|G|}{|G\cdot x|} = \sum_{g\in G} |\mathrm{Fix}(g)|$

and $\displaystyle\sum_{x\in X} \frac{1}{|G\cdot x|} = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|$ (divide both sides by $|G|$)

So it remains to show that the Left-handside is the number of orbits

Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ be a complete list of pairwise distict orbits

Recall these form a partition of X, and $x \in \mathcal{O}_i \Leftrightarrow G\cdot x = \mathcal{O}_i$

$$\sum_{x\in X}\frac{1}{|G\cdot x|} = \sum_{i=1}^{r}\sum_{x\in\mathcal{O}_i}\frac{1}{|G\cdot x|} = \sum_{i=1}^{r}\sum_{x\in\mathcal{O}_i}\frac{1}{|\mathcal{O}_i|}$$

This is $\frac{1}{|\mathcal{O}_i|}$ added to itself $|\mathcal{O}_i|$ times, so equals 1.

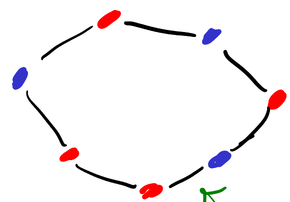
$$= \sum_{i=1}^{r} 1 = r = \#\text{ of orbits}$$

This completes the proof.

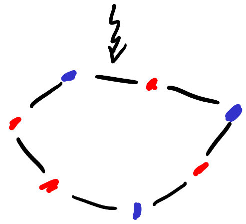
How many necklaces can be made from 4 red beads and 3 blue beads?

To make a necklace (1) Arrange red and blue beads along a string:

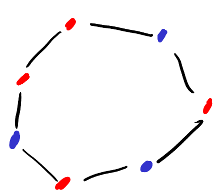
(2) Tie the ends together:



Some different choices in step (1) lead to the same necklace:

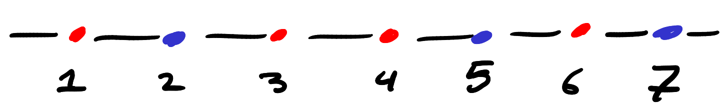


Same necklace, just rotated.

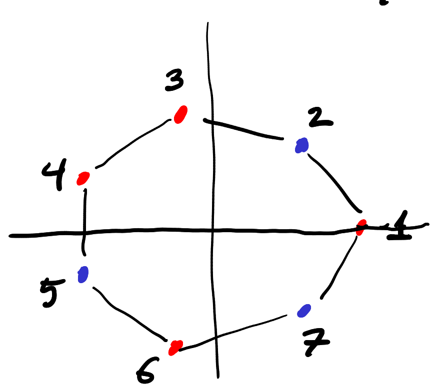


Same, but flipped.

More precise setup: (1) Arrange beads at sites labeled 1–7



(2) Transfer this to the vertices of a regular 7-gon



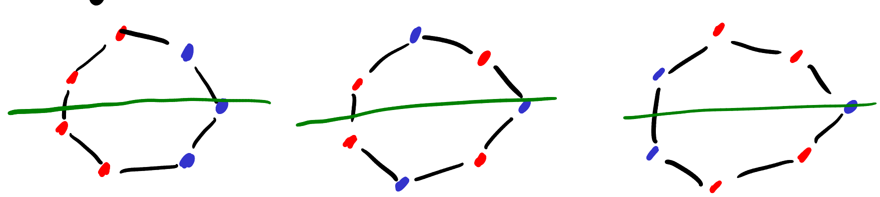
The dihedral group D_7 acts on the set of these pictures, and two necklaces are the same if they lie in the same orbit.
There are $\binom{7}{3} = \frac{7!}{3!4!} = 35$ choices for step(1).

We need to find the number of orbits of D_7 in this set!

$$(\#\text{orbits}) = \frac{1}{|D_7|}\sum_{g\in D_7} |\mathrm{Fix}(g)| = \frac{1}{14}\sum_{g\in D_7}|\mathrm{Fix}(g)|$$

e fixes everything $\quad |\mathrm{Fix}(e)| = 35$
rotation r fixes nothing: always a pair of adjacent red and blue.
rotation r^2 fixes nothing: whatever color 1 is, 3 would have to be same, then 5, then 7, then 2, then 4, then 6, so all would be same color.
Similarly, r^k fixes nothing for $1 \le k \le 6$.
(this may not always be true if you change the numbers of beads)

Now j, the flip about x-axis, fixes 3 things.



each of the 7 flips fixes 3 things, so

$$\#\text{orbits} = \frac{1}{|D_7|}\sum_{g\in D_7}|\mathrm{Fix}(g)| = \frac{1}{14}\Big(\underset{\text{for } e}{35} + \underset{\#\text{flips}}{7}\cdot\underset{\text{for flip}}{3}\Big)$$

$$= \frac{1}{14}(56) = 4$$ There are 4 possible necklaces.