Lecture 18 Orbits and stabilizers

Recall: For a group action $G \times X \to X$ of a group G on a set X:

- The orbit through $x \in X$ is $G \cdot x = \{ g \cdot x \mid g \in G \} \subseteq X$
- The stabilizer of $x \in X$ is $\mathrm{Stab}(x) = \{ g \in G \mid g \cdot x = x \} \leq G$

Lemma 5.1.2: Define a relation $\sim$ on X by

$$x \sim y \iff \exists g \in G \text{ such that } g \cdot x = y.$$

Then $\sim$ is an equivalence relation and the equivalence class of x is the orbit $G \cdot x$.

Proof reflexive: $x \sim x$ since $e \cdot x = x$

symmetric: $x \sim y \Rightarrow \exists g \in G$ s.t. $g \cdot x = y$

then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = e \cdot x = x$

so $y \sim x$

transitive: $x \sim y$ and $y \sim z \Rightarrow \exists g, h \in G$ s.t. $g \cdot x = y$, $h \cdot y = z$

then $(hg) \cdot x = h \cdot (g \cdot x) = h \cdot y = z$ so $x \sim z$.

Equivalence class $[x] = \{ y \mid \exists g \in G \;\; g \cdot x = y \} = \{ g \cdot x \mid g \in G \}$

$= G \cdot x$. ∎

Corollary: The orbits $\{ G \cdot x \mid x \in X \}$ form a partition of X.

Lemma 5.1.12 $\mathrm{Stab}(x)$ is a subgroup of G.

Proof: If $g, h \in \mathrm{Stab}(x)$ then $g \cdot x = x$ $h \cdot x = x$

so $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, so $gh \in \mathrm{Stab}(x)$.

If $g \in \mathrm{Stab}(x)$ $g \cdot x = x$ so $g^{-1} \cdot x = g^{-1} \cdot (g x) = (g^{-1} g) \cdot x = e \cdot x = x$

so $g^{-1} \in \mathrm{Stab}(x)$. ∎

An action is <u>transitive</u> if there is an $x \in X$ such that $G \cdot x = X$. Since the orbits are a partition, this means that $G \cdot y = X$ for <u>every</u> $y \in X$.

The <u>kernel</u> of the action is $\bigcap_{x \in X} \mathrm{Stab}(x)$. It is a subgroup of G.

An action is called <u>faithful</u> (or <u>effective</u>) if its kernel is trivial.

<u>Example</u>: G acts on itself by conjugation $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$.

<u>Orbits</u>: $G \cdot h = \{ghg^{-1} \mid g \in G\}$ is called the <u>conjugacy class of h in G</u>.

<u>Stabilizer</u>:
$$\begin{aligned} \mathrm{Stab}(h) &= \{g \in G \mid ghg^{-1} = h\} = \{g \mid gh = hg\} \\ &= \{g \mid g \text{ commutes with } h\} \\ &= \mathrm{Cent}_G(h) \end{aligned}$$
Called the <u>centralizer of h in G</u>.

<u>kernel</u>:
$$\begin{aligned} \text{kernel} &= \bigcap_{h \in G} \mathrm{Cent}_G(h) = \{g \mid gh = hg \text{ for all } h \in G\} \\ &= Z(G) \text{, the } \underline{\text{center}} \text{ of } G. \end{aligned}$$

Note $G \cdot e = \{geg^{-1} \mid g \in G\} = \{e\}$ so the action is not transitive unless $G = \{e\}$ is trivial.

<u>Example</u> G acts on itself by left multiplication $G \times G \to G$, $(g, h) \mapsto gh$

This is transitive: for any x and y : $y = (yx^{-1}) \cdot x$

The stabilizer is always trivial. $gx = x \Rightarrow g = e$.

<u>Example</u>: Let $H \leq G$ be a subgroup (we do not assume it is normal)
Let $G/H = \{aH \mid a \in G\}$ be the set of left cosets.
Then there is an action of G on G/H:

$$G \times G/H \to G/H \quad , \quad g \cdot (aH) = (ga)H.$$

This action is transitive: $(ba^{-1})(aH) = ba^{-1}aH = bH$ (any $a,b \in H$).

$\mathrm{Stab}(H) = \{g \in G \mid gH = H\} = H$

$\mathrm{Stab}(aH) = \{g \in G \mid gaH = aH\} = \{g \in G \mid a^{-1}gaH = H\}$
$= \{g \mid a^{-1}ga \in H\} = \{g \mid g \in aHa^{-1}\} = aHa^{-1}$

<u>Example</u>: Let G be a group. Let $X = \{H \mid H \leq G\}$ be the set of all subgroups of G. Since conjugation by g is an automorphism of G, it takes subgroups to subgroups.
$c_g(H) = gHg^{-1}$

Define $G \times X \to X \quad g \cdot H = gHg^{-1}$
Then $\mathrm{Stub}(H) = \{g \in G \mid gHg^{-1} = H\} = N_G(H)$
this is called the <u>normalizer</u> of H in G.
Note $\mathrm{Stab}(H) = G \iff H \trianglelefteq G$.

<u>Orbit-Stabilizer theorem</u> (Prop. 5.1.13)
Let $G \times X \to X$ be a group action. Let $x \in X$. Then there is a bijection

$$\psi : G/\mathrm{Stab}(x) \to G \cdot x$$

given by

$$\psi(a\,\mathrm{Stab}(x)) = a \cdot x$$

<u>Proof:</u> Need to check ψ is well-defined.

$$a\,\mathrm{Stab}(x) = b\,\mathrm{Stab}(x) \iff b^{-1}a \in \mathrm{Stab}(x)$$

So if $a\,\mathrm{Stab}(x) = b\,\mathrm{Stab}(x)$, $(b^{-1}a)\cdot x = x$

then $b\cdot x = b\cdot((b^{-1}a)\cdot x) = (bb^{-1}a)\cdot x = a\cdot x$

so the definition is consistent.

For injectivity: if $\psi(a\cdot \mathrm{Stab}(x)) = \psi(b\cdot \mathrm{Stab}(x))$ then $a\cdot x = b\cdot x$

so $b^{-1}a\cdot x = b^{-1}b\cdot x = x$ so $b^{-1}a \in \mathrm{Stab}(x)$

so $a\,\mathrm{Stab}(x) = b\cdot \mathrm{Stab}(x)$.

For surjectivity: Any $y \in G\cdot x$ is $a\cdot x = \psi(a\,\mathrm{Stab}(x))$ for some $a \in G$. $\blacksquare$

<u>Corollary 5.1.14</u> Suppose G is a finite group. Then

$$|G\cdot x| = |G| / |\mathrm{Stab}(x)| \quad \text{and} \quad |G\cdot x| \text{ divides } |G|$$

<u>Proof</u> Orbit-Stabilizer theorem + Lagrange's theorem.

Applications of orbit-stabilizer theorem to combinatories.

① Let $X = \{1, 2, 3, \dots, n\}$. Let $0 \le k \le n$. How many subsets of size k are there in X?

Let $\mathcal{P}_k(X) = \{A \subseteq X \mid |A| = k\}$ be the set of subsets of size k.

The groups S_n acts on X, and so it also acts on $\mathcal{P}_k(X)$.

Eg. $(12) \cdot \{2,3,4\} = \{1,3,4\}$

The action is <u>transitive</u>: any subset of size k can be taken to any other by a permutation.

eg $\{1,2,3\} \longrightarrow \{4,5,6\}$ by $(14)(25)(36)$.

So consider $Y = \{1, 2, \dots, k\} \subseteq X$ then $Y \in \mathcal{P}_k(X)$

By trasitivity, the orbit of Y is everything: $S_n \cdot Y = \mathcal{P}_k(X)$

By the orbit stabilizer theorem, $|\mathcal{P}_k(X)| = |S_n| / |\mathrm{Stab}(Y)|$

Now $|S_n| = n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$

What is $\mathrm{Stab}(Y)$? We have $\sigma \cdot Y = Y$ if σ maps every element in the range $1 \le i \le k$ into the same range. and maps every element $k+1 \le i \le n$ into the same range.

So σ permutes $\{1, 2, \dots, k\}$ and $\{k+1, \dots, n\}$ within themselves.

There are $k!$ ways to permute $\{1, \dots, k\}$

There are $(n-k)!$ ways to permute $\{k+1, \dots, n\}$

Thus $|\text{stab}(Y)| = k!\,(n-k)!$

and so $|\mathcal{P}_k(Y)| = \dfrac{|S_n|}{|\text{stab}(Y)|} = \dfrac{n!}{k!\,(n-k)!}$

This is also known as $\binom{n}{k}$, "n choose k", the binomial coefficient.
It is not really obvious from the formula that $\frac{n!}{k!(n-k)!}$ is always an integer, but our argument proves this.

② How many ways can the letters of MISSISSIPPI be rearranged?
There are 11 letters and S_{11} acts by swapping the letters

$(12) \cdot \text{MISSISSIPPI} = \text{IMSSISSIPPI}$

$(34) \cdot \text{MISSISSIPPI} = \text{MISSISSIPPI}$ (fixed)

S_{11} acts transitively on all arrangements.

So $\#\text{ arrangements} = |S_{11}| / |\text{Stab(one arrangement)}|$

So consider the arrangement SSSSIIII PPM
The stabilizer of this arrangement consist of permutations
that permute the S's $(4! = 24)$
permute the I's $(4! = 24)$
permute the P's $(2! = 2)$
permute the M $(1! = 1)$

so $|\text{Stab(SSSSIIIIPPM)}| = 4!4!2!1! = 1152$
and $\#\text{ arrangements} = 11!/(4!4!2!1!) = 34650$

③ How many strings are there with r_1 objects of one type, r_2 of another, r_3 of a third, and so on up to r_n objects of the nth type?
Answer: $(r_1 + r_2 + \cdots + r_n)!/(r_1!\,r_2!\cdots r_n!)$ (multinomial coefficient)