Lecture 16 Semidirect products

First some terminology. Let G be a group.

An automorphism of G is an isomorphism $\varphi: G \to G$.
The identity function $I: G \to G$ is an automorphism, but there may be others. For instance:

Let $g \in G$. Then $c_g: G \to G$, $c_g(x) = gxg^{-1}$ is called conjugation by g.

Lemma: c_g is an automorphism of G.

Proof $c_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = c_g(x)\,c_g(y)$

The inverse of c_g is $c_{g^{-1}}$: $c_g(c_{g^{-1}}(x)) = g(g^{-1}xg)g^{-1} = x$
$c_{g^{-1}}(c_g(x)) = g^{-1}(gxg^{-1})g = x$.

so c_g is a bijective homomorphism. $\blacksquare$

We denote by $\mathrm{Aut}(G)$ the set of all automorphisms of G.
$\mathrm{Aut}(G)$ is a subset of $\mathrm{Sym}(G) = \{f: G \to G \mid f \text{ is bijective}\}$, and in fact it is a subgroup. (The composition of automorphisms is an automorphism, and the inverse of an automorphism is an automorphism — check this.)

Proposition The function $\alpha: G \to \mathrm{Aut}(G)$, $\alpha(g) = c_g$ is a homomorphism.

<u>Proof</u> $\alpha(g_1g_2)(x) = c_{g_1g_2}(x) = (g_1g_2)x(g_1g_2)^{-1}$
$= g_1g_2xg_2^{-1}g_1^{-1}$

$\alpha(g_1)\circ\alpha(g_2)(x) = c_{g_1}(c_{g_2}(x)) = g_1(g_2xg_2^{-1})g_1^{-1}$
$= g_1g_2xg_2^{-1}g_1^{-1}$

same.

<u>Proposition</u> A subgroup $N \subseteq G$ is normal if and only if $c_g(N) = N$ for all $g \in G$

<u>Proof</u> Suppose $c_g(N) = N$. Take $n \in N, g \in G$. then $gng^{-1} \in c_g(N) = N$ so N is normal.

Suppose N normal. Then $c_g(N) = \{c_g(n) \mid n \in N\} = \{gng^{-1} \mid n \in N\} \subseteq N$ (for any $g \in G$). Then $c_{g^{-1}}(N) \subseteq N$ as well, and

$N = c_g(c_{g^{-1}}(N)) \subseteq c_g(N)$. thus $N = c_g(N)$ ∎

<u>Corollary</u> If $N \trianglelefteq G$, there is a homomorphism $\alpha : G \to \mathrm{Aut}(N)$, $\alpha(g) = (c_g \text{ restricted to } N)$.

<u>Semidirect products</u> Like direct products, we start with two groups A, N (not necessarily subgroups of some third group.) We also choose a homomorphism

$$\alpha : A \to \mathrm{Aut}(N)$$

We then consider the set $N \times A = \{(n,a) \mid n \in N, a \in A\}$ and we define a binary operation.

$$(n,a)(n',a') = (n\cdot\alpha(a)(n'), aa')$$

NB. α is a function $A \to \mathrm{Aut}(N)$ so $\alpha(a) \in \mathrm{Aut}(N)$
so $\alpha(a)$ is a function $N \to N$, and
$\alpha(a)(n')$ is an element of N.

A somewhat nicer notation is to write α_a in place of $\alpha(a)$.
So then the binary operation is written:

$$(n,a)(n',a') = (n\,\alpha_a(n'), aa')$$

<u>Proposition</u> This operation makes the set $N \times A$ into a group.
We denote it by $N \rtimes_\alpha A$.

<u>Proof</u> Associative: $\big((n_1,a_1)(n_2,a_2)\big)(n_3,a_3) = (n_1\alpha_{a_1}(n_2), a_1a_2)(n_3,a_3)$

$$= (n_1\,\alpha_{a_1}(n_2)\,\alpha_{a_1a_2}(n_3),\ a_1a_2a_3)$$

since $\alpha : A \to \mathrm{Aut}(N)$ is homomorphism

$$= (n_1\,\alpha_{a_1}(n_2)\,\alpha_{a_1}(\alpha_{a_2}(n_3)),\ a_1a_2a_3)$$

since $\alpha_{a_1} : N \to N$ is a homomorphism.

$$= (n_1\,\alpha_{a_1}(n_2\alpha_{a_2}(n_3)),\ a_1a_2a_3)$$

$$= (n_1, a_1)(n_2\alpha_{a_2}(n_3),\ a_2a_3)$$

$$= (n_1, a_1)\big((n_2,a_2)(n_3,a_3)\big)$$

Identity: $(e,e)(n,a) = (e\,\alpha_e(n), ea) = (en, ea) = (n,a)$
$(n,a)(e,e) = (n\,\alpha_a(e), ae) = (ne, ae) = (n,a)$

Inverses $(n,a)(\alpha_{a^{-1}}(n^{-1}), a^{-1}) = (n\,\alpha_a\alpha_{a^{-1}}(n^{-1}), aa^{-1})$
$= (n\,\alpha_{aa^{-1}}(n^{-1}), e) = (n\,\alpha_e(n^{-1}), e) = (nn^{-1}, e) = (e,e)$

similarly $(\alpha_{a^{-1}}(n^{-1}), a^{-1})(n,a) = (e,e)$ ∎

How to remember the formula:

$$(n,a)(n',a')$$

$$(n\,\alpha_a(n'),\ a\,a')$$

When the a passes over the n', we apply α_a to n'.

The direct product $N \times A$ is the special case where $\alpha : A \to \mathrm{Aut}(N)$ is trivial: $\alpha_a = (\text{Identity } N \to N)$ for all a.

<u>Properties of $N \rtimes_\alpha A$:</u> Let $\tilde{N} = \{(n,e) \mid n \in N\}$
$\tilde{A} = \{(e,a) \mid a \in A\}$

Then $\tilde{N} \cong N$, $\tilde{A} \cong A$, $\tilde{N}$ and $\tilde{A}$ are subgroups of $N \rtimes_\alpha A$, and $\tilde{N}$ is a normal subgroup of $N \rtimes_\alpha A$, while $\tilde{A}$ is not necessarily normal.
$\tilde{N} \cap \tilde{A} = \{(e,e)\}$, and $\tilde{N}\tilde{A} = N \rtimes_\alpha A$.

Most interesting bit is that $\tilde{N}$ is normal: indeed

$$\begin{aligned}
&(m,b)\,(n,e)\,(\alpha_{b^{-1}}(m^{-1}),b^{-1}) \\
&= (m,b)\,(n\,\alpha_e(\alpha_{b^{-1}}(m^{-1})),\,e b^{-1}) \\
&= (m,b)\,(n\,\alpha_{b^{-1}}(m^{-1}),\,b^{-1}) \\
&= (m\,\alpha_b(n\,\alpha_{b^{-1}}(m^{-1})),\,b b^{-1}) \\
&= (m\,\alpha_b(n)\,m^{-1},\ e) \in \tilde{N}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&(m,b)\,(e,a)\,(\alpha_{b^{-1}}(m^{-1}),b^{-1}) \\
&= (m,b)\,(e\,\alpha_a\alpha_{b^{-1}}(m^{-1}),\,ab^{-1}) \\
&= (\underbrace{m\ \alpha_b\alpha_a\alpha_{b^{-1}}(m^{-1})}_{\text{no reason for this to be } e},\ bab^{-1})
\end{aligned}$$

So $\tilde{A}$ is not necessarily normal.

Recognizing semidirect products

We have the notion of semidirect product that is constructed from the data of two groups N, A, and a homomorphism
$$\alpha : A \to \mathrm{Aut}(N).$$

We want a way to recognize that a given group is isomorphic to a semidirect product $N \rtimes_\alpha A$ for some A, N, α.

Recall: $N \rtimes_\alpha A = \{(n,a) \mid n \in N, a \in A\}$ with operation
$$(n,a)(n',a') = (n\,\alpha_a(n'),\ aa')$$

- There are subgroups $\tilde{N} = \{(n,e) \mid n \in N\}$ $\tilde{A} = \{(e,a) \mid a \in A\}$
- The subgroup $\tilde{N}$ is normal.
- $N \rtimes_\alpha A = \tilde{N}\tilde{A}$ and $\tilde{N} \cap \tilde{A} = \{(e,e)\}$ is the trivial subgroup.

The result is that these properties essentially characterize the semidirect product.

Proposition 3.2.5 Let G be a group, and let N and A be subgroups of G.
Suppose (i) N is normal in G
(ii) $NA = G$
(iii) $N \cap A = \{e\}$

Then G is isomorphic to $N \rtimes_c A$, where $c : A \to \mathrm{Aut}(N)$ is the conjugation homomorphism: $c_a(n) = ana^{-1}$.
More specifically, the function
$$\varphi : N \rtimes_c A \to G \quad \varphi(n,a) = na$$
is an isomorphism.

Remark: the hypothesis (i) is necessary in order to know that $c: A \to \mathrm{Aut}(N)$ makes sense. (If N is not normal then c_a is not necessarily an automorphism of N).

Proof The main thing is to show that $\varphi: N \rtimes_c A \to G$ is a homomorphism.

$$\varphi((n,a)(n',a')) = \varphi(n c_a(n'), aa') = n c_a(n') a a'$$

$$= n a n' a^{-1} a a' = n a n' a' = \varphi(n,a)\varphi(n',a')$$

So it works.

The image of φ is clearly NA, which equals G by hypothesis (ii), so φ is surjective.

Last, $\ker(\varphi) = \{(n,a) \mid na = e\}$. But $na = e$ means $n = a^{-1}$ then $n = a^{-1} \in A$ so $n \in N \cap A = \{e\}$, so $n = e$, and $a = e$ as well. So $\ker(\varphi) = \{(e,e)\}$ is trivial. Thus φ is injective.

This completes the proof that $\varphi: N \rtimes_c A \to G$ is an isomorphism $\blacksquare$