Lecture 15 Direct products

Diamond Isomorphism. If $A \leq G$, $N \trianglelefteq G$, then $AN \leq G$
and $AN/N \cong A/A\cap N$

Example $A = \langle 4 \rangle$ $N = \langle 10 \rangle$ in $G = \mathbb{Z}$
$(A+N)/N = (\langle 4\rangle + \langle 10\rangle)/\langle 10\rangle$
is isomorphic to $A/A\cap N$, and $A\cap N = \langle 4\rangle \cap \langle 10\rangle = \langle 20\rangle$
so $(A+N)/N \cong \langle 4\rangle/\langle 20\rangle$ which is cyclic of order 5 $\cong \mathbb{Z}_5$

Check $A+N = \langle \gcd(4,10)\rangle = \langle 2\rangle$ and $\langle 2\rangle/\langle 10\rangle \cong \mathbb{Z}_5$.

Direct product: Given two groups A and B, we can make
$A\times B = \{(a,b) \mid a\in A, b\in B\}$ into a group:
define $(a,b)(a',b') = (aa', bb')$, where $aa' \in A$ is defined using the given group operation on A, and likewise $bb' \in B$. It is straightforward to check this is an associative operation of $A\times B$, that (e_A, e_B) is an identity element if $e_A \in A$, $e_B \in B$ are identity elements, and that the inverse of (a,b) is (a^{-1}, b^{-1}).

To be clear, $A\times B$ is different from $AB = \{ab \mid a\in A, b\in B\}$
where $A\leq G$ $B \leq G$ are subgroups of the same group G.
However, it is possible that AB and $A\times B$
may be isomorphic.

Eg.. In $G = \mathbb{Z}_2\times\mathbb{Z}_3$, let $A = \{(a,0) \mid a\in\mathbb{Z}_2\}$ $B = \{(0,b) \mid b\in \mathbb{Z}_3\}$
then $AB = G$ and $A \cong \mathbb{Z}_2$ $B \cong \mathbb{Z}_3$ so $AB \cong A\times B$.

But in S_3, $A = \{e, (12)\}$ $B = \{e, (123), (132)\}$ (normal)
then $A \cong \mathbb{Z}_2$, $B \cong \mathbb{Z}_3$, $AB = S_3$ but
S_3 is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$
(since S_3 is not abelian).

<u>Proposition 3.1.5</u> Suppose $A \trianglelefteq G$, $B \trianglelefteq G$, $A \cap B = \{e\}$.
Then $AB \trianglelefteq G$ and
(1) for all $a \in A$ and $b \in B$, $ab = ba$.
(2) $\varphi : A \times B \to AB$, $\varphi(a,b) = ab$, is an isomorphism.

<u>Proof</u> Let $ab \in AB$, $g \in G$. Then

$$g(ab)g^{-1} = \underbrace{(gag^{-1})}_{\substack{\in A \\ \text{since } A \\ \text{normal}}} \underbrace{(gbg^{-1})}_{\substack{\in B \\ \text{since } B \\ \text{normal}}} \in AB. \quad \text{Thus } AB \trianglelefteq G$$

(1) Observe $ab = ba \iff aba^{-1}b^{-1} = e$.
Now

$$aba^{-1}b^{-1} = a \underbrace{(ba^{-1}b^{-1})}_{\substack{\in A \text{ since} \\ A \text{ normal}}} \in A$$

$$||$$

$$\underbrace{(aba^{-1})}_{\substack{\in B \text{ since} \\ B \text{ normal.}}} b^{-1} \in B$$

so $aba^{-1}b^{-1} \in A \cap B = \{e\}$
and $aba^{-1}b^{-1} = e$.

(2) Define $\varphi : A \times B \to AB$ by $\varphi(a,b) = ab$.
φ is surjective by construction.
φ is a homomorphism by (1):
$\varphi((a,b)(a',b')) = \varphi(aa', bb') = aa'bb' = aba'b' = \varphi(a,b)\varphi(a',b')$
Kernel? $\varphi(a,b) = e \Rightarrow ab = e \Rightarrow a = b^{-1}$
$\Rightarrow a \in A \cap B$ and $b \in A \cap B$ so $a = e, b = e$.
kernel is trivial, so φ is injective $\blacksquare$

Example $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with multiplication.

$A = \mathbb{R}_+ = \{ r \in \mathbb{R} \mid r > 0 \}$

$B = \mathcal{U} = \{ e^{i\theta} \mid \theta \in \mathbb{R} \}$

then $A \triangleleft G$, $B \triangleleft G$ since G is abelian.

Also $A \cap B = \{1\}$.

Also $AB = \mathbb{R}_+ \mathcal{U} = \mathbb{C}^*$, because every complex number has a polar form $z = re^{i\theta}$.

So by the proposition, $\varphi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}^*$

$$\varphi(r, e^{i\theta}) = re^{i\theta}$$

is an isomorphism.

We can also think about the direct product of more than two groups. If $A_1, A_2, \ldots, A_n$ are groups then

$$A_1 \times A_2 \times \cdots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \}$$

is a group with operation.

$$(a_1, a_2, \ldots, a_n)(a_1', a_2', \ldots, a_n') = (a_1 a_1', a_2 a_2', \ldots, a_n a_n')$$

(Check it!)

As with the case of two groups, there is a proposition that helps us recognize when a group is isomorphic to a direct product:

Proposition 3.1.12: Suppose $N_1, N_2, \ldots, N_r \triangleleft G$ are normal subgroups and that for all i, $1 \le i \le r$,

$$N_i \cap (N_1 N_2 \cdots N_{i-1} N_{i+1} \cdots N_r) = \{e\}$$

then $N_1 N_2 \cdots N_r \triangleleft G$ and $\varphi : N_1 \times N_2 \times \cdots \times N_r \to N_1 N_2 \cdots N_r$

$$\varphi(n_1, n_2, \ldots, n_r) = n_1 n_2 \cdots n_r$$

is an isomorphism.

Proof: see Goodman.