Lecture 12

Last time introduced two concepts

<u>Equivalence relation</u>: Let X be a set, and let $\sim$ be a binary relation on X. It is an equivalence relation if it is
- reflexive: $\forall x \in X$, $x \sim x$.
- symmetric: $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$.
- transitive: $\forall x, y, z \in X$, $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

<u>Partition</u>: Let X be a set, and let Ω be a set of subsets of X. Ω is a partition if
- $\forall A, B \in \Omega$, either $A \cap B = \emptyset$ or $A = B$.
- $X = \bigcup_{A \in \Omega} A$

Today, the slogan is "equivalence relations are essentially the same as partitions." More precisely, each equivalence relation determines a partition and vice versa.

(A) From equivalence relation to partition:
Let X be a set and let $\sim$ denote an equivalence relation on X. Pick some $x \in X$. Define <u>the equivalence class of x</u> to be the set

$$[x] = \{ y \in X \mid x \sim y \} \subseteq X \quad \text{, a subset of } X.$$

(We sometimes also write $[x]_\sim$ or $[x]_R$ to emphasize dependence on the relation $\sim$ or $R \subseteq X \times X$)

<u>Proposition</u> With notation as above, we have
$\forall x, y \in X$, $x \sim y$ iff $[x] = [y]$.

<u>Proof</u> Suppose $x \sim y$. Take $z \in [x]$. Then $z \sim x$
Now $z \sim x$ and $x \sim y$ implies $z \sim y$ by transitivity
so $z \in [y]$. Thus $[x] \subseteq [y]$.
Also, if $w \in [y]$, $w \sim y$. Now $y \sim x$ by symmetry,
so $w \sim y$ and $y \sim x \Rightarrow w \sim x$, by transitivity.
Thus $[y] \subseteq [x]$ and $[x] = [y]$.

Conversely, suppose $[x] = [y]$. By reflexivity,
$x \sim x$ so $x \in [x]$, and so $x \in [y]$. Thus $x \sim y$.

<u>Corollary</u> With notation as above, we have:
$\forall x, y \in X$, either $[x] \cap [y] = \emptyset$ or $[x] = [y]$

<u>Proof</u>: If $[x] \cap [y] \neq \emptyset$, take $z \in [x] \cap [y]$. Then
$z \in [x]$ so $z \sim x$ and $z \in [y]$ so $z \sim y$.
By symmetry $x \sim z$, which combined with $z \sim y$
yields $x \sim y$. By previous proposition $[x] = [y]$

<u>Corollary</u> Let $\sim$ be an equivalence relation on X.
Then the set of equivalence classes

$$\Omega_\sim = \{ [x]_\sim \mid x \in X \} \text{ is a partition of } X.$$

<u>Proof</u>: Since $x \in [x]$, we see $X = \bigcup_{x \in X} [x] = \bigcup_{A \in \Omega} A$.

The other property of a partition was proved in the previous corollary.

(B) From partitions to equivalence relations.

Now let X be a set, and let Ω be a partition of X.
Define a binary relation $\sim$ on X by

$$x \sim y \quad \text{iff} \quad \exists A \in \Omega \text{ such that } x \in A \text{ and } y \in A.$$

(x and y lie in same part of the partition)

__Proposition__ Let Ω be a partition of X, and let $\sim$ be defined as above. Then $\sim$ is an equivalence relation.

__Proof__ For reflexivity, since $X = \bigcup_{A \in \Omega} A$, for any $x \in X$, there is some $A \in \Omega$ such that $x \in A$. Since $x \in A$ and $x \in A$, $x \sim x$.

For symmetry, suppose $x \sim y$, meaning $\exists A \in \Omega$ such that $x \in A$ and $y \in A$. Then $y \in A$ and $x \in A$, so $y \sim x$.

For transitivity, suppose $x \sim y$ and $y \sim z$. This means
$\exists A \in \Omega$ such that $x \in A$ and $y \in A$ and
$\exists B \in \Omega$ such that $y \in B$ and $z \in B$.
Notice that $y \in A$ and $y \in B$, so $y \in A \cap B$, and $A \cap B \neq \emptyset$.
Thus $A = B$ (property of a partition), so $z \in B = A$.
Now $x \in A$ and $z \in A$, so $x \sim z$.

__Proposition__ If Ω is a partition of X, and $\sim_\Omega$ is the equivalence relation defined from Ω, then the equivalance classes of $\sim_\Omega$ are precisely the elements of Ω.

$$\{ [x]_{\sim_\Omega} \mid x \in X \} = \Omega.$$

<u>Proof</u> Take $x \in X$. Since Ω is a partition, there is a unique $A \in \Omega$ such that $x \in A$.

Then $x \sim_\Omega y \iff y \in A$ so

$$[x]_{\sim_\Omega} = \{ y \in X \mid x \sim_\Omega y \} = \{ y \in X \mid y \in A \} = A$$

Thus $[x]_{\sim_\Omega} \in \Omega$, so each equiv. class is in Ω.

Also, if $A \in \Omega$, then $A = [z]_{\sim_\Omega}$ for any $z \in A$, so A is an equiv. class

<u>Proposition</u>: Let $\sim$ be an equivalence relation on X, let $\Omega = \{ [x]_\sim \mid x \in X \}$ be the partition into equivalence classes. Further, let $\sim_\Omega$ be the equivalence relation constructed from Ω. Then $\sim \, = \, \sim_\Omega$, i.e. $\forall x, y \in X \quad x \sim y \iff x \sim_\Omega y$.

<u>Proof</u>: Suppose $x \sim y$ (original equiv. rel.)

then $[x]_\sim = [y]_\sim$, so $x, y \in [x]_\sim = [y]_\sim \in \Omega$,

so $x \sim_\Omega y$.

Conversely, if $x \sim_\Omega y$, then there is some $A \in \Omega$ such that $x, y \in A$. This $A = [z]_\sim$ for some $z \in X$

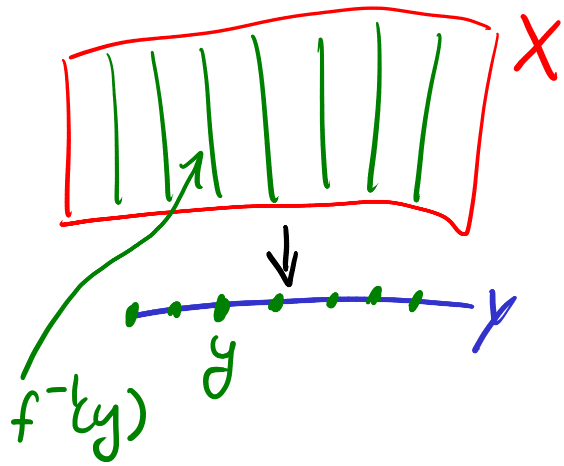
$$\text{so} \begin{cases} x \in [z]_\sim \\ y \in [z]_\sim \end{cases} \Rightarrow \begin{cases} x \sim z \\ y \sim z \end{cases} \Rightarrow x \sim y. \quad \blacksquare$$

Restatement: Each equivalence relation on X determines a partition on X, and each partition of X is determined by a unique equivalence relation. There is a bijective correspondence

$$\{ \text{equivalence relations on } X \} \longleftrightarrow \{ \text{partitions of } X \}$$

An important example of equivalence relation/partition.

Let X and Y be sets and let $f: X \to Y$ be a function.
for each $y \in Y$, consider $f^{-1}(y) = \{x \in X \mid f(x) = y\}$, the set of preimages of y. We also call this set the <u>fiber</u> of f at y.



If we set $\Omega = \{f^{-1}(y) \mid y \in Y\}$, then Ω is a partition of X.

The corresponding equivalence relation is

$$x \sim_f y \iff f(x) = f(y).$$

Every equivalence relation/partition also arises this way:

Let $\sim$ be an equivalence relation with set of equivalence classes $\Omega = \{[x] \mid x \in X\}$. Define a function.

$$\pi : X \to \Omega \qquad \pi(x) = [x].$$

Observe that π is surjective and

$$\pi(x) = \pi(y) \iff [x] = [y] \iff x \sim y.$$

Thus $\pi^{-1}([x]) = \{y \in X \mid \pi(y) = [x]\} = \{y \in X \mid y \sim x\} = [x]$

$$\pi^{-1}(\underbrace{[x]}_{\text{element of } \Omega}) = \underbrace{[x]}_{\text{subset of } X.}$$

We call π the "canonical projection".

<u>Example</u>: G a group, $H \leq G$ a subgroup.

Let $\Omega = \{aH \mid a \in G\}$ be the set of left cosets.

<u>Notation</u> In this case, we write G/H "G mod H" for the set of left cosets. $G/H = \Omega$ above.

The function $\pi : G \to G/H$ $\pi(a) = aH$ is the canonical projection or <u>quotient map</u> of G onto G/H.

<u>Example</u> G a group. Say $a \in G$ is conjugate to $b \in G$ if $\exists g \in G$ such that $b = gag^{-1}$.

Say $a \sim b$ if a is conjugate to b. This is an equivalence relation.

Reflexive: since $a = eae^{-1}$, we have $a \sim a$.

Symmetric: if $b = gag^{-1}$, then $a = g^{-1}bg = hbh^{-1}$ with $h = g^{-1}$ so $a \sim b \Rightarrow b \sim a$.

Transitive: if $b = gag^{-1}$ and $c = hbh^{-1}$ then
$$c = hbh^{-1} = h(gag^{-1})h^{-1} = (hg)a(g^{-1}h^{-1}) = (hg)a(hg)^{-1}$$
so $a \sim b$ and $b \sim c \Rightarrow a \sim c$.

In this context, the equivalence classes are called the <u>conjugacy classes</u> in G.

<u>Ex</u> In S_3, conjugacy classes are $\{e\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\}$.

<u>Ex</u> if G is abelian (commutative: $ab = ba$). Then $gag^{-1} = gg^{-1}a = a$ for any $a, g \in G$. We can then see conjugacy classes are $\{\{a\} \mid a \in G\}$. (singleton sets).