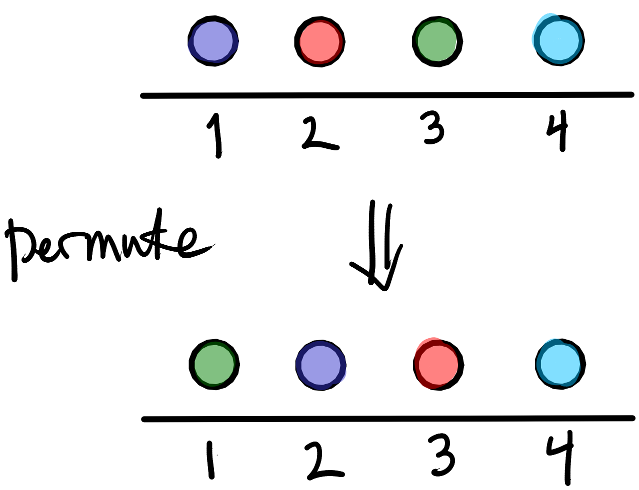
417 Lecture 2

Last lecture: the set of symmetries of $R \subset \mathbb{R}^3$ forms a "group." Now look at another fundamental example.

2) Permutations: reordering a set of objects.



$1 \mapsto 2$
$2 \mapsto 3$
$3 \mapsto 1$
$4 \mapsto 4$

Notation: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

Definition: A permutation of a finite set F is a bijection $\pi : F \to F$.

Recall: A function $f : X \to Y$ is a bijection
$\iff$ there is an inverse function $f^{-1} : Y \to X$

$\iff$ f is both injective (one-to-one) and surjective (onto)

i) f is injective iff $(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

ii) f is surjective iff $(\forall y \in Y, \exists x \in X \text{ s.t. } f(x) = y)$

Useful fact: When $F = X = Y$ is a $\boxed{\text{finite}}$ set, a function $\pi : F \to F$ is bijective $\Leftrightarrow$ it is surjective $\Leftrightarrow$ it is injective.

For any finite set F, we can always number the elements $1, 2, \ldots, n = |F|$. In studying permutations we might as well assume that

$$F = \{1, 2, 3, \ldots, n\}.$$

Then a notation for a bijection $\pi : F \to F$ is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} \quad \text{where } a_i \in \{1, 2, \ldots, n\}$$

This means that $\pi(1) = a_1$, $\pi(2) = a_2$, $\pi(i) = a_i$ and so on.

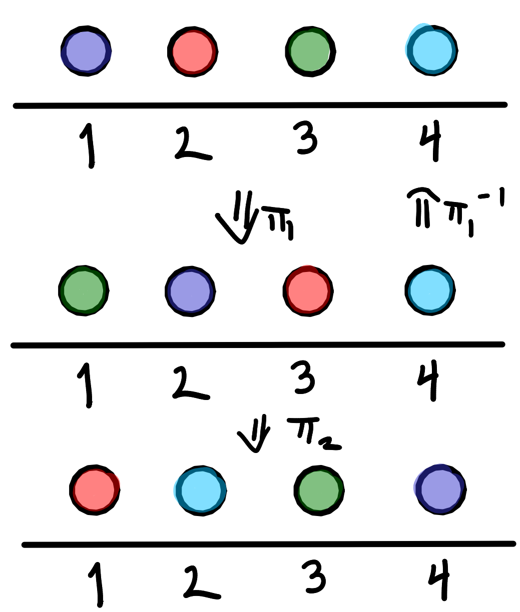
<u>Fact</u>: Composition of bijections is a bijection. (composition of permutations is a permutation).

<u>Example</u>:

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \pi_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\pi_2 \circ \pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$



Cycle notation: if $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$,
we could write $1 \mapsto 2 \mapsto 3$ a cycle of length 3.

We write (123) for this cycle.
This is another notation for permutations.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (123)(4) = (123) = (231) = (312)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24) = (31)(42) = (24)(13)$$

Notation is not unique!

$$\pi_2 \circ \pi_1 = (13)(24)(123) = (142)(3) = (142) \Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

Another such problem: $n = 5$.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (12345)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = (124)(35)$$

$$\pi_2 \circ \pi_1 = (124)(35)(12345) = (143)(25) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

$$\pi_1 \circ \pi_2 = (12345)(124)(35) = (13)(254) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

Composition of permutations is not commutative!
$\pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1$ sometimes.

Definition: Denote by S_n the set of permutations of $F = \{1, 2, \dots, n\}$.

Lemma: The number of elements of S_n is $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.

Proof: $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$

n choices for a_1,
$n-1$ choices for a_2,
$n-2$ choices for a_3,
$\vdots$
1 choice for a_n.

In total we have $n(n-1) \cdot (n-2) \cdots 2 \cdot 1$ possibilities.

Example: S_3 has 6 elements.

$$S_3 = \{ I, (12), (13), (23), (123), (321) \}$$

Note:
$(21) = (12)$ $\quad$ $(123) = (231) = (312)$
$(31) = (13)$ $\quad$ $(321) = (213) = (132)$
$(32) = (23)$

Special types of permutations in S_n:

(i) Identity permutation: $I=(1)(2)\cdots(n)$ does nothing

(ii) Transposition: $\pi=(ab)$ $a \mapsto b \mapsto a$
Swaps a and b and that's all.

(iii) Cycle of length k: $\pi=(a_1 a_2 \cdots a_k)$
(k-cycle)

$$a_1 \mapsto a_2 \mapsto a_3 \mapsto \cdots \mapsto a_k \mapsto a_1$$

cyclically permutes $a_1, a_2, \ldots, a_k$ and that's all.

Note: 1-cycle = Identity
2-cycle = transposition.

Example: In S_3, there are only cycles.
In S_4, there are other elements:
$(13)(24)$, $(14)(23)$, $(12)(34)$
In S_5, can also have
$(12)(345)$, etc.

Consider two cycles $(a_1 a_2 \cdots a_k)$, $(b_1 b_2 \cdots b_\ell)$
They are disjoint if none of the a's equals any of the b's.

Eg. (123) and (456) are disjoint
(142) and (35) are disjoint
(123) and (345) are not disjoint (b/c 3)

Proposition 1: Every $\pi \in S_n$ can be written as a product of disjoint cycles, in a way that is <u>essentially unique</u> (unique up to order of the factors).

Proof: $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$

Look at sequence $1, \pi(1), \pi^2(1) = \pi(\pi(1)), \pi^3(1), \ldots$
eventually the sequence comes back to 1:
$\pi^k(1) = 1$ (choose smallest such k)
So π contains the k-cycle $(1\ \pi(1)\ \pi^2(1) \cdots \pi^{k-1}(1))$

Next choose some $a \in \{1, \ldots, n\}$ that does not appear so far, and consider $a, \pi(a), \pi^2(a), \ldots$
Eventually this comes back to a: $\pi^\ell(a) = a$.
So π contains the ℓ-cycle $(a\ \pi(a)\ \pi^2(a) \cdots \pi^{\ell-1}(a))$

Keep repeating this process until all elements $a \in \{1, \ldots, n\}$ have been accounted for. ∎

<u>Example</u>: $\underbrace{(124)(5432)}_{\text{Not disjoint}} = \underbrace{(125)(34)}_{\text{disjoint.}}$

<u>Note</u>: If π_1 and π_2 are disjoint cycles, then $\pi_1 \pi_2 = \pi_2 \pi_1$ (they commute)

e.g. $(125)(34) = (34)(125)$

Proposition 2: Every $\pi \in S_n$ can be written as a product of transposition (in several ways). For a given π, the number of transpositions appearing in such a factorization is always either even or odd.

Proof of first part: π can be written as a product of cycles by Prop. 1, so we just need to show that a cycle can be written as a product of transpositions.
Look at:

$$(a_1 a_2 \cdots a_k) = (a_{k-1} a_k)(a_{k-2} a_k) \cdots (a_2 a_k)(a_1 a_k)$$

Proof of second part is omitted. $\boxtimes$

Example:

$$\pi = (124)(5432) = \underbrace{(24)(14)(32)(42)(52)}_{5 \text{ transp.}}$$

$$= (125)(34) = \underbrace{(25)(15)(34)}_{3 \text{ transp.}}$$

odd Permutation.

$$\pi = (125)(324) = \underbrace{(25)(15)(24)(34)}_{4 \text{ transp.}}$$

even permutation.

Definition: the sign of a permutation π is

$$\operatorname{sgn}(\pi) = \begin{cases} +1 & \pi \text{ is even.} \\ -1 & \pi \text{ is odd.} \end{cases}$$

Example: $\operatorname{sgn}(I) = 1 \qquad \operatorname{sgn}((ab)) = -1$

$$\operatorname{sgn}\big((a_1 a_2 \cdots a_k)\big) = (-1)^{k-1}$$

If $\pi = (a_1 \cdots a_k)$ is a cycle, the inverse is $\pi^{-1} = (a_k \, a_{k-1} \cdots a_2 \, a_1)$.

Eg. $\pi = (4215) \Rightarrow \pi^{-1} = (5124) = (1245)$

If π is a product of cycles, then π^{-1} is the product of the inverse cycles in the reverse order.

$$\pi = (a_1 \cdots a_k)(b_1 \cdots b_\ell) \cdots (z_1 \cdots z_r)$$
$$\pi^{-1} = (z_r \cdots z_1) \cdots (b_\ell \cdots b_1)(a_k \cdots a_1)$$

$$\pi \circ \pi^{-1} = (a_1 \cdots a_k)(b_1 \cdots b_\ell) \cdots (z_1 \cdots z_r)(z_r \cdots z_1) \cdots (b_\ell \cdots b_1)(a_k \cdots a_1)$$

$$= I$$

Thus S_n is a group. $S_n = \{\text{permutations of } \{1, \dots, n\}\}$ operation = composition, which is

(i) Associative: $(\pi_1 \circ \pi_2) \circ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi_3)$
(ii) has Identity: $\pi \circ I = I \circ \pi = \pi$
(iii) has Inverses: $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = I$.