Math 417: Homework 9

Due Friday, November 10, 2023

1. Let G be a group, and let $A \leq G$ be a subgroup. Suppose that there is a homomorphism $\pi : G \to A$ with the following property: for all $a \in A$, $\pi(a) = a$. Prove that G is isomorphic to a semidirect product of $\ker(\pi)$ and A, as in
$$G \cong \ker(\pi) \rtimes_\alpha A,$$
where $\alpha : A \to \mathrm{Aut}(\ker(\pi))$ is given by $\alpha_a(g) = aga^{-1}$.

2. Goodman, exercise 3.2.2.

3. Goodman, exercise 3.2.4.

4. Goodman, exercise 3.2.6.

5. Let p be a prime number. Prove that there is a nonabelian group with $p^2 - p$ elements. *Hint:* First prove the following fact: There is a nontrivial homomorphism
$$\alpha : \mathbb{Z}_p^\times \to \mathrm{Aut}(\mathbb{Z}_p), \quad \alpha_{[k]}([x]) = [kx],$$
where $\mathbb{Z}_p$ is a group under addition, and $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus \{[0]\}$ is a group under multiplication.

6. Goodman, exercise 5.1.4.

7. Goodman, exercise 5.1.5.

8. Goodman, exercise 5.1.6.