Math 417: Homework 8

Due Friday, October 27, 2023

1. **(10 points)** An *affine transformation* of the vector space $\mathbb{R}^n$ is a transformation of the form

$$T_{A,b} : \mathbb{R}^n \to \mathbb{R}^n, \quad T_{A,b}(x) = Ax + b,$$

where A is an invertible $n \times n$ matrix, and b is a vector. The set of all affine transformations is denoted $\mathrm{Aff}(n)$:

$$\mathrm{Aff}(n) = \{T_{A,b} : A \in \mathrm{GL}(n,\mathbb{R}),\ b \in \mathbb{R}^n\}.$$

   (a) Prove that $\mathrm{Aff}(n)$ is a group under composition of functions.

   (b) Prove that the function

$$\Phi : \mathrm{Aff}(n) \to \mathrm{GL}(n,\mathbb{R}), \quad \Phi(T_{A,b}) = A$$

   is a surjective homomorphism.

   (c) Let N be the kernel of Φ. Prove that N is isomorphic to the vector space $\mathbb{R}^n$ (where the group operation on $\mathbb{R}^n$ is vector addition).

   (d) Deduce that $\mathrm{Aff}(n)/N$ is isomorphic to $\mathrm{GL}(n,\mathbb{R})$.

2. **(5 points)** Goodman, exercise 2.7.4.

3. **(5 points)** Let $G = \mathbb{Z}_8$, and consider the subgroups $H = \langle [2] \rangle$ and $K = \langle [4] \rangle$. Note that $K \le H \le G$.

   (a) List the cosets of H in G, showing the elements in each coset.

   (b) List the cosets of K in G, showing the elements in each coset.

   (c) List the cosets of K in H, showing the elements in each coset.

   (d) List the cosets of H/K in G/K, showing the elements in each coset.

   (e) Write down the correspondence between G/H and $(G/K)/(H/K)$ that is guaranteed by Proposition 2.7.14.

4. **(5 points)** Goodman, exercise 2.7.11.

5. **(5 points)** Goodman, exercise 3.1.9.

6. **(10 points)** $\mathbb{R}$ is group of real numbers (with addition), and $\mathbb{C}^*$ is the group of nonzero complex numbers (with multiplication).

   Prove that there is an isomorphism

$$\mathbb{C}^* \cong \mathbb{R} \times (\mathbb{R}/\mathbb{Z}).$$

   Hint: Think about exponentials and logarithms of complex numbers.