Math 417: Homework 7

Due Friday, October 20, 2021

1. **(5 points)** Goodman, exercise 2.7.2.

2. **(10 points)** Goodman, exercise 2.7.6.

3. **(5 points)** Goodman, exercise 2.7.7.

4. **(10 points)** Let G be a group and let $x, y \in G$. The expression $xyx^{-1}y^{-1}$ is called the *commutator* of x and y, because $xyx^{-1}y^{-1} = e$ if and only if $xy = yx$ ("two elements commute iff their commutator is trivial").

   Let S be the set of all commutators in G:

   $$S = \{xyx^{-1}y^{-1} : x, y \in G\}.$$

   Let C be the subgroup generated by S: $C = \langle S \rangle$. Then C is called the *commutator subgroup* of G. (Remark: the set S is not necessarily a subgroup of G.)

   (a) Prove that if ϕ is an automorphism of G, then $\phi(S) = S$ and $\phi(C) = C$.

   (b) Prove that C is a normal subgroup of G. *Hint:* Conjugation by g is an automorphism.

   (c) Prove that G/C is abelian.

   (d) Prove that if H is a normal subgroup of G such that G/H is abelian, then $C \subseteq H$.

5. **(10 points)** Let G be the group of 2×2 matrices that are upper triangular and invertible, namely:

   $$G = \left\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} : x, y, z \in \mathbb{R}, x \neq 0, y \neq 0 \right\}.$$

   (You make take for granted that this set is a group under matrix multiplication.) There is a subgroup $H \leq G$ comprising those matrices in G that have ones on the diagonal, namely:

   $$H = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{R} \right\}.$$

   Prove that

   (a) H is a normal subgroup of G,

   (b) H is abelian,

   (c) G/H is abelian, and

   (d) G is not abelian.