

Math 417: Homework 4

Due Friday, September 22, 2023

1. Goodman, exercise 2.1.6.
2. Goodman, exercise 2.1.7.

The next three exercises ask you to prove the classification of groups with four elements: The group \mathbb{Z}_4 (with addition) has four elements, as does the group $G_{\text{rectangle}}$ of rotational symmetries of the rectangle (whose multiplication table is given in Figure 1.3.4).

The **classification of groups with four elements** states the following:

- The groups \mathbb{Z}_4 and $G_{\text{rectangle}}$ are not isomorphic to each other, and
 - if H is any group with four elements, then H is isomorphic either to \mathbb{Z}_4 or to $G_{\text{rectangle}}$.
3. Goodman, exercise 2.1.5. (This exercise asks you to prove the first bullet point.)
 4. Goodman, exercise 2.1.10. (This exercise is a lemma for the next one.)
 5. Goodman, exercise 2.1.11. (This exercise asks to you prove the second bullet point.)
 6. Goodman, exercise 2.2.2.
 7. Goodman, exercise 2.2.6.
 8. Let a be an element of a group such that $a^k = e$ for some $k > 0$. Let n be the least positive integer such that $a^n = e$. Show that the elements $e, a, a^2, \dots, a^{n-1}$ are all distinct.
Note: This is one of the steps in the used in the proof of the classification of cyclic groups, but it was not proved in the text nor in the lecture notes. Since the goal of this exercise is to complete that proof, you may not use Proposition 2.2.17 or 2.2.20 in your solution.
 9. In the permutation group S_n , suppose that σ_1 is a k -cycle, σ_2 is an ℓ -cycle, and σ_1 and σ_2 are disjoint cycles. Find the order of the product $\sigma_1\sigma_2$ in terms of k and ℓ , and justify your answer.